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Linear potential theory of steady internal supersonic flow with quasi-cylindrical geometry. Part 1. Flow in ducts

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Based on linear potential theory, the general three-dimensional problem of steady supersonic flow inside quasi-cylindrical ducts is formulated as an initial-boundary-value problem for the wave equation, whose general solution arises as an infinite double series of the Fourier–Bessel type. For a broad class of solutions including the general axisymmetric case, it is shown that the presence of a discontinuity in wall slope leads to a periodic singularity pattern associated with non-uniform convergence of the corresponding series solutions, which thus are unsuitable for direct numerical computation. This practical difficulty is overcome by extending a classical analytical method, viz. Kummer's series transformation. A variety of elementary flow fields is presented, whose complex cellular structure can be qualitatively explained by asymptotic laws governing the propagation of small perturbations on characteristic surfaces.

1. Introduction

Because of the continuing interest in high-speed aerodynamics, the theory of external supersonic flow past a body of given geometry is well-developed today. A great variety of two- and three-dimensional problems has been solved analytically and provides the basis for a comprehensive understanding of the underlying physical mechanisms. In contrast, the theory of internal supersonic flow, i.e. the flow in the interior of a duct of given contour (or in a free jet, which can be considered as a limiting case where the guiding surface is formed by the flow itself), is in a much more unsatisfactory state. This is especially the case for flows with cylindrical geometry, which are of importance in many applications such as supersonic intakes, wind tunnels and jet engines.

Based on a famous paper by Prandtl (1904) on supersonic free jets, von Kármán (1907) was the first to notice that linear potential theory should also be able to describe supersonic flow inside a quasi-cylindrical duct; however, he did not carry out any calculations. Apparently, the first attempts in this direction are due to Ward (1945, 1948), who applied an operational method to obtain a formal series solution for the special case of uniform parallel flow entering an axisymmetric duct with linearly varying cross-section. He noticed that the resulting flow field contains periodically distributed singularities (discontinuities and logarithmic poles), which are associated with physical phenomena that linear theory is unable to describe adequately, viz. shocks and expansion waves. Unfortunately, these singularities give rise to non-uniform convergence and thus to oscillatory behaviour of the partial sums of the corresponding infinite series ('Gibbs' phenomenon'), which does not vanish no matter how many terms are included into the summation. Thus, the seemingly trivial task of computing the flow field from the formal solution turns out to be a formidable problem, which indeed has not yet been solved.

Mack (1947) and later Ludloff & Reiche (1949) developed a supersonic 'source–sink' method similar to that devised by von Kármán & Moore (1932) for the external flow past slender bodies of revolution. They calculated formal solutions corresponding to a constant and linearly increasing axisymmetric source distribution on the surface of a 'skeleton cylinder' and discusssed the possible duct shapes that a superposition of such flows may produce. Some aspects of the general problem were studied in a subsequent paper by Kolodner (1950). In none of these works however, were the obtained series solutions suitable for direct numerical computation because of non-uniform convergence caused by singularities and thus for the same reasons as in the case of Ward's solution.

Presumably because of these apparent practical shortcomings, the interest in linear theory decreased and no significant contribution has been published since 1950. There is no reported attempt to extend the theory to the general case of a quasi-cylindrical duct of arbitrary geometry and inflow conditions; and quantitative results for the few known solutions are still not available. Although various research papers, particularly in the closely related field of linear acoustics, discuss the general singular behaviour of non-analytic series solutions of the wave equation for the case of cylindrical geometry, they almost completely neglect their actual computation. There has also been no attempt to apply alternative methods of solving the wave equation as for example geometrical acoustics (cf. Friedlander 1958), which however, although it circumvents the series representation of singular functions, is of considerable mathematical complexity and does not allow a general solution. Consequently, problems of cylindrical supersonic duct flow are today tackled rather by numerical methods, even in cases where linear potential theory should apply.

Thus, it must be concluded that the linear theory of cylindrical supersonic duct flow is far from being complete and that its capability to describe real flow has remained unexplored until the present day. The present paper therefore aims towards filling this gap. In §2, the solution for the general case of arbitrary geometry and inflow conditions, which is not available so far, is derived by the eigenfunction method. The mathematical reasons for non-analytic behaviour and their physical meaning are examined in §3, and by extending a classical analytical tool, viz. Kummer's series transformation, the problems associated with the numerical computation of non-uniformly converging solutions are completely resolved. Thus, the predictions of linear potential theory are made accessible to *quantitative* evaluation. In order also to provide a qualitative understanding of the underlying physical mechanisms, asymptotic laws governing the propagation of small perturbations will be derived in §4. Finally, some elementary examples of axisymmetric and non-axisymmetric flow fields will be discussed in detail in §5. Besides Ward's classical solution, this discussion also includes several cases for which exact solutions were not previously known.

2. General theory

2.1. *Mathematical formulation*

Consider the supersonic flow of a compressible medium inside a duct whose contour deviates only slightly from that of a right circular cylinder of radius R_0 with the contour variations in axial and azimuthal direction being small everywhere. Consequently, the flow can be thought of being composed of a uniform cylindrical flow parallel to the duct axis and small superimposed perturbations caused by the deviation of the duct surface from ideal cylindrical geometry or by inflow conditions deviating from ideal parallel flow. Since the distortion of the streamlines is small, the strength of possibly occurring shocks in the flow will be weak. If furthermore viscosity and heat conduction are neglected, then the flow can be assumed to be irrotational and isentropic. Hence, linear potential flow theory can be applied.

Take a right-handed system of Cartesian coordinates (x, y, z) such that the z-axis is along the duct axis and $z = 0$ at the mouth of the tube, and let w_0 be the velocity of the undisturbed flow in the z-direction with the Mach number $M_0 > 1$. It is well established in linear potential flow theory (see for example Ward 1955), that in non-dimensional cylindrical polar coordinates (r, φ, ζ), which are connected with the dimensional Cartesian coordinates (x, y, z) via

$$x = rR_0 \cos\varphi, \tag{2.1a}$$

$$y = rR_0 \sin\varphi, \tag{2.1b}$$

$$z = \zeta R_0 (M_0^2 - 1)^{1/2}, \tag{2.1c}$$

the scalar velocity potential $\phi(r, \varphi, \zeta)$ of small irrotational perturbations of a uniform parallel flow in the ζ-direction obeys the well-known wave equation:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \varphi^2} - \frac{\partial^2 \phi}{\partial \zeta^2} = 0. \tag{2.2}$$

The velocity perturbation vector is given by the gradient of the scalar velocity potential ϕ and, consequently, the velocity components u, v, w in the (r, φ, ζ)-system are given by

$$\frac{u}{w_0} = \frac{\partial \phi}{\partial r}, \quad \frac{v}{w_0} = \frac{1}{r}\frac{\partial \phi}{\partial \varphi}, \quad \frac{w}{w_0} = 1 + \frac{1}{(M_0^2 - 1)^{1/2}}\frac{\partial \phi}{\partial \zeta}. \tag{2.3a, b, c}$$

If furthermore the fluid under consideration is assumed to be a perfect gas with constant ratio κ of the specific heats c_p and c_v, then the thermodynamic variables of state – pressure p, density ρ, and temperature T – are connected with the velocity potential ϕ by its first derivative with respect to ζ, i.e. with the axial component of the velocity perturbation:

$$\frac{p}{p_0} = 1 - \frac{\kappa M_0^2}{(M_0^2 - 1)^{1/2}}\frac{\partial \phi}{\partial \zeta}, \tag{2.4a}$$

$$\frac{\rho}{\rho_0} = 1 - \frac{M_0^2}{(M_0^2 - 1)^{1/2}}\frac{\partial \phi}{\partial \zeta}, \tag{2.4b}$$

$$\frac{T}{T_0} = 1 - \frac{(\kappa - 1) M_0^2}{(M_0^2 - 1)^{1/2}}\frac{\partial \phi}{\partial \zeta}, \tag{2.4c}$$

where p_0, ρ_0 and T_0 denote pressure, density and temperature in the undisturbed parallel flow, respectively. Thus, the theoretical treatment of the complete flow field is reduced to the solution of the wave equation (2.2), for which appropriate boundary and initial conditions must now be specified.

Let the contour $R(\varphi, \zeta)$ of the duct at polar angle φ and distance ζ from the mouth be given by

$$R(\varphi, \zeta) = R_0 (1 + \lambda(\varphi, \zeta)) \tag{2.5}$$

with the dimensionless contour function $\lambda(\varphi, \zeta)$ describing the deviation of the duct shape from ideal cylindrical geometry. Both $\lambda(\varphi, \zeta)$ as well as its first derivatives are *a priori* assumed to be small compared to unity. In consequence of the neglect of viscosity, the physical boundary condition of perfect slip must be satisfied at the wall, i.e. the flow has to be tangential to the surface of the duct. Mathematically, this means that the scalar product of the velocity vector with the surface normal has to vanish. By using the elementary methods of differential geometry, this condition is easily formulated, and upon neglecting squares and higher powers of small quantities, we obtain the linearized boundary condition to be satisfied by the velocity potential $\phi(r, \varphi, \zeta)$ on the surface of the semi-infinite cylinder $r = 1$, $\zeta > 0$:

$$\left.\frac{\partial \phi}{\partial r}\right|_{r=1} = \frac{1}{(M_0^2 - 1)^{1/2}} \frac{\partial \lambda}{\partial \zeta}. \tag{2.6}$$

In order to complete the Cauchy problem for the wave equation (2.2), we further have to demand that both the velocity potential ϕ and its first derivative with respect to ζ are *known* functions $f(r, \varphi)$ and $g(r, \varphi)$ respectively at $\zeta = 0$, i.e. at the entry of the duct:

$$\left.\phi\right|_{\zeta=0} = f(r, \varphi), \quad \left.\frac{\partial \phi}{\partial \zeta}\right|_{\zeta=0} = g(r, \varphi). \tag{2.7a,b}$$

From (2.3) and (2.4), the physical meaning of the initial conditions (2.7) is obvious. While $g(r, \varphi)$ describes the perturbation of *axial* velocity and therefore of pressure, density and temperature at the entry of the duct, $f(r, \varphi)$ corresponds to the perturbations of the velocity components *perpendicular* to the tube axis and therefore describes deviations from coaxial inflow into the duct.

Hence, we have reduced the problem of determining the complete internal flow in ducts of arbitrary quasi-cylindrical geometry to the solution of the linear wave equation (2.2) under the inhomogeneous boundary condition (2.6) and the initial conditions (2.7). Once the velocity potential ϕ is determined for a particular problem, all physical flow variables of interest can immediately be determined by (2.3) and (2.4), which relate velocity, pressure, density and temperature to the first derivatives of the potential function.

Note that the above formulation of steady supersonic flow in cylindrical ducts as an initial-boundary-value problem for the wave equation (2.2) allows the construction of a variety of analogies to other physical phenomena governed by the same equation. For example, as has been pointed out by Kolodner (1950), the problem under study is identical with that of a vibrating circular membrane with prescribed normal derivative at the boundary. Another noteworthy analogy, which appears less artificial with respect to the boundary condition, can be constructed by identifying the velocity potential ϕ with the vertical surface displacement of a shallow liquid pool moving under the influence of gravity in a cylindrical containment (cf. Lamb 1975). In this case, the inhomogeneity of the boundary condition (2.6) (i.e. the axial duct slope) corresponds to the negative radial acceleration of the container walls and, thus, the *steady* compressible flow problem becomes mathematically identical with an *unsteady* hydrodynamic problem, viz. the oscillations of a liquid pool inside a cylindrical containment with its walls being in accelerated motion perpendicular to its axis.

2.2. *General solution*

As pointed out above, our task consists now in finding the general solution of the wave equation (2.2) under the inhomogeneous boundary condition (2.6) and the initial conditions specified by (2.7). It is well established in mathematical physics that, for solving a partial differential equation under an inhomogeneous boundary condition, the eigenfunctions and eigenvalues of the associated homogeneous problem can be used (see for example Courant & Hilbert 1931). In the present case, this means that the general solution of our problem is an infinite linear combination of the eigenfunctions of the wave equation (2.2) in cylindrical coordinates with the eigenvalues determined by the homogeneous form of the boundary condition (2.6), viz.

$$\left.\frac{\partial \phi}{\partial r}\right|_{r=1} = 0. \tag{2.8}$$

Thus, by further requiring the velocity potential and its derivatives to be continous on the axis $r = 0$ except at isolated singular points, the most general form of the velocity potential $\phi(r, \varphi, \zeta)$ can be written as

$$\phi(r,\varphi,\zeta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\beta'_{mn} r)\ [A_{mn}(\zeta)\cos m\varphi + B_{mn}(\zeta)\sin m\varphi], \tag{2.9}$$

where $J_m(x)$ are the Bessel functions of first kind and integer order m and the β'_{mn} denote the real *non-negative* zeros of the first derivative $J'_m(x)$ of $J_m(x)$ arranged in ascending order of magnitude (note that since $J'_0(x) = -J_1(x)$, this definition means $\beta'_{01} = 0$):

$$J'_m(\beta'_{mn}) = 0. \tag{2.10}$$

In order to determine the unknown coefficient functions $A_{mn}(\zeta)$, $B_{mn}(\zeta)$ in (2.9), we adapt a method described by Tolstov (1976) for the solution of the equation of heat conduction under inhomogeneous boundary conditions. By multiplying the solution ansatz (2.9) with $J_\mu(\beta'_{\mu\nu} r)\,\mathrm{e}^{\mathrm{i}\mu\varphi}$ and integrating over the unit circle, all terms of the infinite sum are vanishing except those with coinciding indices $m = \mu$, $n = \nu$ because of the orthogonality of the eigenfunctions. Thus, by using complex notation for convenience, we obtain the general relation

$$\begin{aligned} C_{mn}(\zeta) &:= A_{mn}(\zeta) + \mathrm{i}\,B_{mn}(\zeta) \\ &= \frac{2\beta'^2_{mn}}{\pi\,(\beta'^2_{mn} - m^2)\,J^2_m(\beta'_{mn})} \int_0^{2\pi}\!\!\int_0^1 r\,\phi(r,\varphi,\zeta)\,J_m(\beta'_{mn} r)\,\mathrm{e}^{\mathrm{i}m\varphi}\,\mathrm{d}r\mathrm{d}\varphi. \end{aligned} \tag{2.11}$$

For $m = 0$, the right-hand side of (2.11) has to be divided by 2. Now, by applying integration by parts twice with respect to r and φ, respectively, we obtain from (2.11)

$$\begin{aligned} C_{mn}(\zeta) = \frac{2}{\pi\,(\beta'^2_{mn} - m^2)\,J^2_m(\beta'_{mn})} \Bigg\{ & J_m(\beta'_{mn}) \int_0^{2\pi} \left.\frac{\partial\phi}{\partial r}\right|_{r=1} \mathrm{e}^{\mathrm{i}m\varphi}\,\mathrm{d}\varphi \\ & - \int_0^{2\pi}\!\!\int_0^1 r \left[\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2}\right] J_m(\beta'_{mn} r)\,\mathrm{e}^{\mathrm{i}m\varphi}\,\mathrm{d}r\mathrm{d}\varphi \Bigg\}, \end{aligned} \tag{2.12}$$

while on the other hand, by differentiating (2.11) twice with respect to ζ, we have ($C''_{mn}(\zeta)$ denotes $\mathrm{d}^2 C_{mn}/\mathrm{d}\zeta^2$)

$$C''_{mn}(\zeta) = \frac{2\beta'^2_{mn}}{\pi\,(\beta'^2_{mn} - m^2)\,J^2_m(\beta'_{mn})} \int_0^{2\pi}\!\!\int_0^1 r\,\frac{\partial^2\phi}{\partial\zeta^2}\,J_m(\beta'_{mn} r)\,\mathrm{e}^{\mathrm{i}m\varphi}\,\mathrm{d}r\mathrm{d}\varphi. \tag{2.13}$$

Hence, upon multiplying (2.12) by β'^2_{mn}, adding the resulting equation to (2.13) and using both the wave equation (2.2) and the inhomogeneous boundary condition (2.6), we finally obtain the following ordinary differential equation for the complex coefficient function $C_{mn}(\zeta) = A_{mn}(\zeta) + \mathrm{i}\, B_{mn}(\zeta)$:

$$C''_{mn} + \beta'^2_{mn} C_{mn} = \frac{1}{(M_0^2-1)^{1/2}} \frac{2\beta'^2_{mn}}{\pi(\beta'^2_{mn}-m^2)\, J_m(\beta'_{mn})} \int\limits_0^{2\pi} \frac{\partial\lambda}{\partial\zeta}\, \mathrm{e}^{\mathrm{i}m\varphi}\, \mathrm{d}\varphi, \tag{2.14}$$

where as above, the right-hand side has to be divided by 2 for $m = 0$. Thus, the coefficient functions $A_{mn}(\zeta)$, $B_{mn}(\zeta)$ of the velocity potential (2.9) are given as solutions of the well-known ordinary differential equation for a mass–spring system with external exciting force, the eigenfrequency being the zero β'_{mn} of $J'_m(x)$ and the exciting force being essentially determined by the axial slope of the duct. The solution of (2.14) for any given contour function $\lambda(\varphi, \zeta)$ can be calculated by standard analytical methods (e.g. Laplace transformation) and is therefore not a serious problem. The initial conditions (2.7) are related to the initial values $C_{mn}(0)$, $C'_{mn}(0)$ required for the solution of (2.14) by taking (2.11) and its first ζ-derivative, respectively, at $\zeta = 0$. Thus, by (2.9), (2.11) and (2.14), the general problem can be considered as formally solved.

2.3. *Solutions with harmonic azimuthal dependence*

In order to simplify the further discussion without significant loss of generality, we will restrict ourselves in the following to problems where the azimuthal dependence of the velocity potential is given by the *single* harmonic function $\cos m\varphi$ ($m = 0, 1, 2\ldots$). Consequently, the contour function $\lambda(\varphi, \zeta)$ must be of the form

$$\lambda(\varphi, \zeta) = \lambda_m(\zeta) \cos m\varphi\,, \tag{2.15}$$

while the velocity potential (2.9) simplifies to

$$\phi(r, \varphi, \zeta) = \cos m\varphi \sum_{n=1}^{\infty} A_{mn}(\zeta)\, J_m(\beta'_{mn} r) = \phi_m(r, \zeta) \cos m\varphi\,. \tag{2.16}$$

The coefficient functions $A_{mn}(\zeta)$ are determined by the simplified form of the differential equation (2.14):

$$A''_{mn} + \beta'^2_{mn} A_{mn} = \frac{1}{(M_0^2-1)^{1/2}} \frac{2\beta'^2_{mn}}{(\beta'^2_{mn}-m^2)\, J_m(\beta'_{mn})} \lambda'_m(\zeta), \tag{2.17}$$

where $\lambda'_m(\zeta)$ denotes $\mathrm{d}\lambda_m/\mathrm{d}\zeta$. Finally, the integral relation (2.11) simplifies to

$$A_{mn}(\zeta) = \frac{2\beta'^2_{mn}}{(\beta'^2_{mn}-m^2)\, J_m^2(\beta'_{mn})} \int\limits_0^1 r\, \phi_m(r, \zeta)\, J_m(\beta'_{mn} r)\, \mathrm{d}r\,, \tag{2.18}$$

both equations (2.17) and (2.18) being valid for all m without division by 2 for $m = 0$. By applying Laplace transformation to the ordinary differential equation (2.17), solving for the transform of $A_{mn}(\zeta)$ and inverting by use of the convolution theorem, we obtain

$$A_{mn}(\zeta) = A_{mn}(0)\cos\beta'_{mn}\zeta + \frac{1}{\beta'_{mn}} A'_{mn}(0)\sin\beta'_{mn}\zeta$$
$$+ \frac{1}{(M_0^2-1)^{1/2}} \frac{2\beta'_{mn}}{(\beta'^2_{mn}-m^2)\,J_m(\beta'_{mn})} \int_0^\zeta \sin\beta'_{mn}(\zeta-\tau)\,\lambda'_m(\tau)\,\mathrm{d}\tau, \qquad (2.19)$$

with the initial values $A_{mn}(0)$, $A'_{mn}(0)$, which can be related to ϕ_m and $\partial\phi_m/\partial\zeta$ at $\zeta = 0$ via the integral relation (2.18) and its first derivative with respect to ζ, respectively

$$A_{mn}(0) = \frac{2\beta'^2_{mn}}{(\beta'^2_{mn}-m^2)\,J_m^2(\beta'_{mn})} \int_0^1 r\,\phi_m\big|_{\zeta=0} J_m(\beta'_{mn}r)\,\mathrm{d}r, \qquad (2.20a)$$

$$A'_{mn}(0) = \frac{2\beta'^2_{mn}}{(\beta'^2_{mn}-m^2)\,J_m^2(\beta'_{mn})} \int_0^1 r\,\frac{\partial\phi_m}{\partial\zeta}\bigg|_{\zeta=0} J_m(\beta'_{mn}r)\,\mathrm{d}r. \qquad (2.20b)$$

Hence, the solution ansatz (2.16) with its coefficient functions $A_{mn}(\zeta)$ given by (2.19) and (2.20) represents the general form of the velocity potential $\phi(r,\varphi,\zeta)$ for the case of harmonic azimuthal dependence $\cos m\varphi$. Similar relations are obtained for the case of sinusoidal azimuthal dependence $\sin m\varphi$ and, thus, the most general solution (2.9) can be considered as a harmonic Fourier series in φ with each coefficient being itself an infinite series of the same type as $\phi_m(r,\zeta)$ in (2.16).

3. The singularities of internal supersonic flow in cylindrical ducts

3.1. *Conditions for the occurrence of singularities*

In his first investigations on internal supersonic flow in axisymmetric ducts, Ward (1945) pointed out that linear potential theory may provide solutions which exhibit mathematical singularities at certain locations in the flow field. From a physical point of view, these singularities are representatives of phenomena that linear potential theory fails to describe adequately, e.g. shock waves and expansion fans. Furthermore, the occurrence of singularities in functions described by infinite series gives rise to non-uniform convergence and thus to oscillatory behaviour of the partial sums ('Gibbs' phenomenon'), which seriously hampers the proper numerical evaluation of these series. Therefore, the aim of the following considerations is to discuss the singular behaviour of the solutions provided by linear potential theory and to devise a general method for the practical evaluation of the corresponding non-uniformly converging series.

As already mentioned, we will restrict ourselves to the case of harmonic azimuthal dependence of the velocity potential $\phi(r,\varphi,\zeta)$ as described by (2.16), where the radial and axial dependence is given by an infinite series of the form

$$\phi_m = \sum_{n=1}^{\infty} A_{mn}\,J_m(\beta'_{mn}r). \qquad (3.1)$$

Series of this type are well known in the mathematical literature as special cases of Dini series or Fourier–Bessel series of the second kind (cf. Watson 1944; Tolstov 1976). The singular behaviour of such series is related to the asymptotic form of their coefficients in a quite similar manner as is the case for Fourier series (cf. Lighthill

1970). It is elementary to show (Tolstov 1976), that for sufficiently large n, the condition

$$|A_{mn}| \leqslant \frac{L}{\beta'^{p+\varepsilon}_{mn}} \tag{3.2}$$

with $0 < \varepsilon < 1$ and L being some positive constant is sufficient to guarantee uniform convergence of the Dini series (3.1) and its first $p-1$ derivatives with respect to r in $[0,1]$. Consequently, in order to discuss the singularities of the supersonic flow fields under study, we have to examine the general asymptotic behaviour for large n of the three integrals involved in the relations (2.19),(2.20) for the coefficients $A_{mn}(\zeta)$.

Asymptotic expansions of integrals can be derived by appropriate methods described in the literature, e.g. in the textbook by Bleistein & Handelsman (1986). The methods which apply to the integrals contained in (2.19),(2.20) are described in chapter 6 (especially section 6.3) of this book; the corresponding calculations are straightforward but rather lengthy. Therefore, only the results are presented here. For the convolution integral in (2.19), we obtain under the assumption that $\lambda'_m(\zeta)$ is continuous for $\zeta > 0$

$$\int_0^\zeta \sin \beta'_{mn}(\zeta-\tau)\,\lambda'_m(\tau)\,\mathrm{d}\tau \sim \frac{\lambda'_m(\zeta) - \lambda'_m(+0)\cos\beta'_{mn}\zeta}{\beta'_{mn}} + O\left(\frac{1}{\beta'^2_{mn}}\right), \tag{3.3}$$

while for the coefficient integrals (2.20), which are both of the same type, we get

$$\int_0^1 r\,\chi(r)\,J_m(\beta'_{mn}r)\,\mathrm{d}r \sim \frac{J_m(\beta'_{mn})}{\beta'^2_{mn}}\left.\frac{\mathrm{d}\chi}{\mathrm{d}r}\right|_{r=1} + O\left(\frac{1}{(\beta'_{mn})^{7/2}}\right), \tag{3.4}$$

where $\chi(r)$ is a function representing either $\phi_m(r,\zeta)$ or its first derivative with respect to ζ at $\zeta = 0$. (On deriving (3.4), it has further been assumed that $\chi(r)$ is continuous in $[0,1]$ and that $\chi(0) = 0$ except in the axisymmetric case $m = 0$, where finite values are allowed at $r = 0$.) By substituting (3.3), (3.4) into (2.19), (2.20) and considering (2.15), (2.6) and the inequality

$$|J_m(\beta'_{mn})| \geqslant \frac{L}{(\beta'_{mn})^{1/2}} \tag{3.5}$$

with L being some positive constant (Tolstov 1976), we obtain the following asymptotic expression for the coefficient functions $A_{mn}(\zeta)$ of the velocity potential (2.16):

$$\begin{aligned} A_{mn}(\zeta) = {} & \frac{\lambda'_m(0) - \lambda'_m(+0)}{(M_0^2-1)^{1/2}} \frac{2\cos\beta'_{mn}\zeta}{(\beta'^2_{mn}-m^2)\,J_m(\beta'_{mn})} \\ & + \frac{\lambda'_m(\zeta)}{(M_0^2-1)^{1/2}} \frac{2}{(\beta'^2_{mn}-m^2)\,J_m(\beta'_{mn})} + \frac{r_m(\beta'_{mn}\zeta)}{(\beta'_{mn})^{5/2}}, \end{aligned} \tag{3.6}$$

where $r_m(x)$ is some bounded function with bounded first derivative for $x \to \infty$, whose exact form depends on the particular initial and boundary conditions and is not of interest here. If now (3.6) is substituted into the ansatz (2.16) for the velocity potential, each term on the right-hand side will produce a Dini series of the same type as (3.1).

Since the third term in (3.6) varies in magnitude as $(\beta'_{mn})^{-5/2}$, it follows immediately from (3.2) that both the corresponding Dini series

$$\sum \frac{r_m(\beta'_{mn}\zeta)}{(\beta'_{mn})^{5/2}} J_m(\beta'_{mn} r) \tag{3.7}$$

as well as its first derivatives converge uniformly and, thus, represent bounded continuous functions throughout the field.

By using the inequality (3.5), we obtain the result that the second term in (3.6) varies in magnitude as $(\beta'_{mn})^{-3/2}$. Thus, the corresponding series expression

$$\frac{\lambda'_m(\zeta)}{(M_0^2-1)^{1/2}} \sum \frac{2\,J_m(\beta'_{mn} r)}{(\beta'^2_{mn}-m^2)\,J_m(\beta'_{mn})} \tag{3.8}$$

converges uniformly, but not its first derivative with respect to r. The non-uniform convergence is a natural result of the fact that by prescribing the inhomogeneous boundary condition (2.6), we have demanded a *non-vanishing* first derivative $\partial\phi/\partial r$ of the velocity potential at $r = 1$, although upon differentiation of the general solution (2.9) with respect to r, all terms of the resulting infinite series formally *vanish* at $r = 1$ so that a discontinuity must necessarily arise at this location. Therefore, the non-uniform convergence is merely due to 'mathematical embarassment' and not to singularities corresponding to real physical effects. Indeed, it can be shown from (2.18), that the Dini series in (3.8) can be expressed in closed analytical form:

$$\sum_{n=2}^{\infty} \frac{2\,J_0(\beta'_{0n} r)}{\beta'^2_{0n}\,J_0(\beta'_{0n})} = \tfrac{1}{2}\left(r^2-\tfrac{1}{2}\right), \tag{3.9a}$$

$$\sum_{n=1}^{\infty} \frac{2\,J_m(\beta'_{mn} r)}{(\beta'^2_{mn}-m^2)\,J_m(\beta'_{mn})} = \frac{1}{m} r^m, \qquad m > 0, \tag{3.9b}$$

where the functions on the right-hand sides and all their derivatives are bounded and continuous on $[0, 1]$. (Note that for $m = 0$, the summation has to start at $n = 2$, since $\beta'_{01} = 0$.) Consequently, the second term on the right-hand side of (3.6) also corresponds to a bounded continuous function with bounded continuous first derivatives throughout the field, provided that $\mathrm{d}^2\lambda_m/\mathrm{d}\zeta^2$ is continuous for $\zeta > 0$.

Like the second term, the first term on the right-hand side of (3.6) also varies as $(\beta'_{mn})^{-3/2}$. Thus, the corresponding series expression

$$\frac{\lambda'_m(0)-\lambda'_m(+0)}{(M_0^2-1)^{1/2}} \sum \frac{2\,J_m(\beta'_{mn} r)\cos\beta'_{mn}\zeta}{(\beta'^2_{mn}-m^2)\,J_m(\beta'_{mn})} \tag{3.10}$$

converges uniformly, but neither its first derivative with respect to r nor its first derivative with respect to ζ converge uniformly. As obvious from the prefactor, terms of the form (3.10) only arise if $\lambda'_m(0) \neq \lambda'_m(+0)$, i.e. if there is a discontinuity in the slope of the boundary streamline at $\zeta = 0$. This result strongly suggests that singularities should be present in the flow field, since it is known that physically the abrupt bending of streamlines implies the occurrence of shocks or expansion waves in supersonic flow. Unfortunately, it does not seem possible to sum (3.10) to a closed expression which would allow an immediate discussion of its singularities.

However, the results obtained so far allow at least the separation of the bounded continuous parts of *any* given solution (2.16) from the part which presumably contains singularities of physical relevance, viz. (3.10). By writing the Dini series (3.10) in the form

$$S_m(r,\zeta) := \sum_{n=1}^{\infty} a_{mn}(\zeta)\, J_m(\beta'_{mn} r)\,, \tag{3.11}$$

where in order to avoid an undetermined expression at $m = 0$, $n = 1$, the coefficients $a_{mn}(\zeta)$ are defined by

$$a_{mn}(\zeta) = \begin{cases} 0 & m = 0, n = 1 \\ \dfrac{2\cos\beta'_{mn}\zeta}{(\beta'^2_{mn} - m^2)\, J_m(\beta'_{mn})} & \text{otherwise}, \end{cases} \tag{3.12}$$

we obtain for any Dini Series (3.1) by suitable addition and subtraction of (3.12) the following general expression:

$$\phi_m(r,\zeta) = \Delta\lambda'\, S_m(r,\zeta) + \sum_{n=1}^{\infty} \left(A_{mn}(\zeta) - \Delta\lambda'\, a_{mn}(\zeta)\right)\, J_m(\beta'_{mn} r)\,, \tag{3.13}$$

where for brevity, $\Delta\lambda'$ denotes

$$\Delta\lambda' = \frac{\lambda'_m(0) - \lambda'_m(+0)}{(M_0^2 - 1)^{1/2}}\,. \tag{3.14}$$

From the previous discussion, it is obvious that the continuous bounded part of the solution is now represented by the explicit sum on the right-hand side of (3.13), while any singularities which may occur upon abrupt bending of the streamlines at $\zeta = 0$ (i.e. $\Delta\lambda' = 0$) are contained in the first derivatives of $S_m(r,\zeta)$. Hence, we have shifted both the general discussion of the singularities and the practical problems arising from non-uniform convergence to the evaluation of the first derivatives of $S_m(r,\zeta)$.

Note that in this subsection, only the case of a discontinuity of wall slope at $\zeta = 0$ has been considered, but the analysis may be repeated with minor modifications to determine the singularities associated with other types of discontinuity in the wall. In particular, for discontinuous *curvature*, all velocity components are continuous and the results given here apply directly to the axial *gradients* of the velocity components, while the results for the radial gradients can be obtained in a similar manner. In addition, since the problem is linear, all results also hold for discontinuities located at any other point of the wall.

3.2. *The evaluation of the first derivatives of* $S_m(r,\zeta)$

By the considerations of the previous subsection, we have reduced the general discussion of the singularities in supersonic duct flow and the evaluation of the corresponding non-uniformly converging series to the evaluation of the first derivatives of the series $S_m(r,\zeta)$, whose universal character has thus become obvious. From the mathematical literature, several analytical methods are known for the evaluation of non-uniformly converging infinite series. One of these methods and probably the best candidate for handling such series is an algebraic transformation which was found by E.E. Kummer in 1837 (cf. Knopp 1928). The basic idea behind Kummer's series transformation is to subtract from a given series a suitably chosen one, the so-called comparison series, which has about the same poor rate of convergence, but whose sum can be determined in closed analytical form. To fix notation, suppose that the given series is

$$S = \sum_{n=1}^{\infty} s_n \tag{3.15}$$

and suppose that we know the sum

$$\widetilde{S} = \sum_{n=1}^{\infty} \tilde{s}_n , \tag{3.16}$$

where s_n and $\tilde{s}_n$ are asymptotically equal in the following manner:

$$s_n = \tilde{s}_n + O\left(\frac{1}{n^{1+\varepsilon}}\right), \qquad \varepsilon > 0. \tag{3.17}$$

Then, by simply subtracting (3.16) from (3.15), the unknown sum S can be written as

$$S = \widetilde{S} + \sum_{n=1}^{\infty} (s_n - \tilde{s}_n) = \widetilde{S} + R. \tag{3.18}$$

Thus, we are reduced to computing the remainder series R, which by (3.17) converges uniformly and thus approaches a continuous function rather rapidly, while the singularities of the given series (3.15) are now buried in the closed expression $\widetilde{S}$ for the comparison series (3.16). Therefore, the singularities of S can be discussed by discussing the singularities of $\widetilde{S}$, while the evaluation of the remainder series R is not hampered by non-uniform convergence any more.

Hence, by adopting the notation introduced in (3.15)–(3.18) and writing the first derivatives of $S_m(r, \zeta)$ in the form

$$S_{m,r}(r,\zeta) := \frac{\partial S_m(r,\zeta)}{\partial r} = \sum_{n=1}^{\infty} s_{mn,r}(r,\zeta), \tag{3.19a}$$

$$S_{m,\zeta}(r,\zeta) := \frac{\partial S_m(r,\zeta)}{\partial \zeta} = \sum_{n=1}^{\infty} s_{mn,\zeta}(r,\zeta), \tag{3.19b}$$

where $s_{mn,r}(r,\zeta)$, $s_{mn,\zeta}(r,\zeta)$ denote the derivatives of the terms of (3.11)

$$s_{mn,r}(r,\zeta) = \begin{cases} 0 & m = 0, n = 1 \\ \dfrac{2\beta'_{mn}\, J'_m(\beta'_{mn} r)\, \cos \beta'_{mn}\zeta}{(\beta'^2_{mn} - m^2)\, J_m(\beta'_{mn})} & \text{otherwise}, \end{cases} \tag{3.20a}$$

$$s_{mn,\zeta}(r,\zeta) = \begin{cases} 0 & m = 0, n = 1 \\ -\dfrac{2\beta'_{mn}\, J_m(\beta'_{mn} r)\, \sin \beta'_{mn}\zeta}{(\beta'^2_{mn} - m^2)\, J_m(\beta'_{mn})} & \text{otherwise}, \end{cases} \tag{3.20b}$$

our problem consists in finding appropriate comparison terms $\tilde{s}_{mn,r}(r,\zeta)$, $\tilde{s}_{mn,\zeta}(r,\zeta)$ which have the desired asymptotic behaviour (3.17) and, furthermore, allow the analytical summation of closed expressions $\widetilde{S}_{m,r}(r,\zeta)$, $\widetilde{S}_{m,\zeta}(r,\zeta)$.

Suitable comparison series can be obtained by replacing the Bessel functions, their derivatives and the roots of the derivatives by their asymptotic expansions. For the Bessel function $J_m(x)$ and its derivative $J'_m(x)$, the following asymptotic expansions are valid for large arguments x (Abramowitz & Stegun 1972):

$$J_m(x) \sim \left(\frac{2}{\pi x}\right)^{1/2} \cos(x - m\pi/2 - \pi/4) + O\left(\frac{1}{x^{3/2}}\right), \tag{3.21a}$$

$$J'_m(x) \sim -\left(\frac{2}{\pi x}\right)^{1/2} \sin(x - m\pi/2 - \pi/4) + O\left(\frac{1}{x^{3/2}}\right), \tag{3.21b}$$

while the roots β'_{mn} of $J'_m(x)$ have an asymptotic expansion for large n:

$$\beta'_{mn} \sim \mu'_{mn} + O\left(\frac{1}{\mu'_{mn}}\right), \qquad \mu'_{mn} := (n + m/2 - 3/4)\pi. \tag{3.21c}$$

By substituting (3.21) into (3.20), we obtain the following asymptotic expansions, which are valid for large n:

$$s_{mn,r}(r,\zeta) \sim -\frac{1}{r^{1/2}}\left[\frac{\sin\mu'_{mn}(r+\zeta-1)}{\mu'_{mn}} + \frac{\sin\mu'_{mn}(r-\zeta-1)}{\mu'_{mn}}\right] + O\left(\frac{1}{n^2}\right), \tag{3.22a}$$

$$s_{mn,\zeta}(r,\zeta) \sim -\frac{1}{r^{1/2}}\left[\frac{\sin\mu'_{mn}(r+\zeta-1)}{\mu'_{mn}} - \frac{\sin\mu'_{mn}(r-\zeta-1)}{\mu'_{mn}}\right] + O\left(\frac{1}{n^2}\right), \tag{3.22b}$$

provided that $r > 0$. When $r = 0$, r must be put equal to zero *before* introducing the asymptotic expansions (3.21), and we get

$$s_{mn,r}(0,\zeta) \sim \begin{cases} -\left(\dfrac{\pi}{2}\right)^{1/2} (-1)^n \dfrac{\cos\mu'_{1n}\zeta}{(\mu'_{1n})^{1/2}} + O\left(\dfrac{1}{n^{3/2}}\right) & m = 1 \\ 0 & \text{otherwise}, \end{cases} \tag{3.23a}$$

$$s_{mn,\zeta}(0,\zeta) \sim \begin{cases} (2\pi)^{1/2} (-1)^n \dfrac{\sin\mu'_{0n}\zeta}{(\mu'_{0n})^{1/2}} + O\left(\dfrac{1}{n^{3/2}}\right) & m = 0 \\ 0 & \text{otherwise}, \end{cases} \tag{3.23b}$$

since $J_m(0) = 0$ for all $m \neq 0$ and $J'_m(0) = 0$ for all $m \neq 1$. Obviously, the asymptotic expansions (3.22), (3.23) satisfy (3.17), and thus their leading terms can be used as comparison terms for the derivatives (3.19) of $S_m(r,\zeta)$.

Hence, by introducing the notation

$$\Psi_m(x) = \sum_{n=1}^{\infty} \psi_{mn}(x) = \sum_{n=1}^{\infty} \frac{\sin\mu'_{mn}x}{\mu'_{mn}}, \tag{3.24a}$$

$$\Omega_0(x) = \sum_{n=1}^{\infty} \omega_{0n}(x) = \sum_{n=1}^{\infty} (-1)^n \frac{\sin\mu'_{0n}x}{(\mu'_{0n})^{1/2}}, \tag{3.24b}$$

$$\Omega_1(x) = \sum_{n=1}^{\infty} \omega_{1n}(x) = \sum_{n=1}^{\infty} (-1)^n \frac{\cos\mu'_{1n}x}{(\mu'_{1n})^{1/2}}, \tag{3.24c}$$

we can write the comparison terms $\tilde{s}_{mn,r}(r,\zeta)$, $\tilde{s}_{mn,\zeta}(r,\zeta)$ in the form

$$\tilde{s}_{mn,r}(r,\zeta) = \begin{cases} -r^{-1/2}\,[\psi_{mn}(r+\zeta-1) + \psi_{mn}(r-\zeta-1)] & r > 0 \\ -(\pi/2)^{1/2}\,\omega_{1n}(\zeta) & r = 0,\, m = 1 \\ 0 & \text{otherwise}, \end{cases} \quad (3.25a)$$

$$\tilde{s}_{mn,\zeta}(r,\zeta) = \begin{cases} -r^{-1/2}\,[\psi_{mn}(r+\zeta-1) - \psi_{mn}(r-\zeta-1)] & r > 0 \\ (2\pi)^{1/2}\,\omega_{0n}(\zeta) & r = 0,\, m = 0 \\ 0 & \text{otherwise}, \end{cases} \quad (3.25b)$$

while the comparison sums $\widetilde{S}_{m,r}(r,\zeta)$, $\widetilde{S}_{m,\zeta}(r,\zeta)$ can be expressed as

$$\widetilde{S}_{m,r}(r,\zeta) = \begin{cases} -r^{-1/2}\,[\Psi_m(r+\zeta-1) + \Psi_m(r-\zeta-1)] & r > 0 \\ -(\pi/2)^{1/2}\,\Omega_1(\zeta) & r = 0,\, m = 1 \\ 0 & \text{otherwise}, \end{cases} \quad (3.26a)$$

$$\widetilde{S}_{m,\zeta}(r,\zeta) = \begin{cases} -r^{-1/2}\,[\Psi_m(r+\zeta-1) - \Psi_m(r-\zeta-1)] & r > 0 \\ (2\pi)^{1/2}\,\Omega_0(\zeta) & r = 0,\, m = 0 \\ 0 & \text{otherwise}, \end{cases} \quad (3.26b)$$

and are thus completely related to the simple Fourier series (3.24), whose closed sums remain to be determined. As has been shown by Dillmann & Grabitz (1994), Fourier series of a similar type, which arise in connection with comparison series for Fourier–Bessel series of the first kind, can be summed analytically by use of Lerch's transcendent function. This method also applies in the present case and yields the following closed expressions for the Fourier series (3.24*a*):

$$\Psi_m(x) = \begin{cases} -\displaystyle\sum_{n=1}^{m/2} \frac{\sin \mu'_{0n} x}{\mu'_{0n}} + \frac{1}{2}\,\mathrm{sgn}\,\sin\frac{\pi x}{4} - \frac{1}{2\pi}\ln\frac{1-\sin \pi x/4}{1+\sin \pi x/4} & m \text{ even}, \\ -\displaystyle\sum_{n=1}^{(m-1)/2} \frac{\sin \mu'_{1n} x}{\mu'_{1n}} + \frac{1}{2}\,\mathrm{sgn}\,\sin\frac{\pi x}{4} + \frac{1}{2\pi}\ln\frac{1-\sin \pi x/4}{1+\sin \pi x/4} & m \text{ odd}, \end{cases} \quad (3.27)$$

where $\mathrm{sgn}\,x$ denotes the signum function, which gives $-1, 0$ or $+1$ depending on whether x is negative, zero or positive. As is obvious from (3.27), $\Psi_m(x)$ is a periodic function with period 8, which has discontinuities of magnitude 1 at $x = 0, \pm 4, \pm 8 \ldots$ due to the signum function and logarithmic poles of alternating sign at $x = \pm 2, \pm 6, \pm 10 \ldots$. A graph of $\Psi_m(x)$ is given for $m = 0$ and $m = 1$ in figure 1 (note that for higher values of m, $\Psi_m(x)$ differs from these cases only by a bounded function, since the sums in (3.27) are finite). For the two Fourier series (3.24*b*) and (3.24*c*), we obtain the analytical sums

$$\Omega_0(x) = \frac{1}{2(2\pi)^{1/2}}\left[\mathrm{Z}\left(\frac{3+x}{8}\right) - \mathrm{Z}\left(\frac{3-x}{8}\right) - \mathrm{Z}\left(\frac{7+x}{8}\right) + \mathrm{Z}\left(\frac{7-x}{8}\right)\right], \quad (3.28)$$

$$\Omega_1(x) = \frac{1}{2(2\pi)^{1/2}}\left[\mathrm{Z}\left(\frac{7+x}{8}\right) + \mathrm{Z}\left(\frac{7-x}{8}\right) - \mathrm{Z}\left(\frac{3+x}{8}\right) - \mathrm{Z}\left(\frac{3-x}{8}\right)\right], \quad (3.29)$$

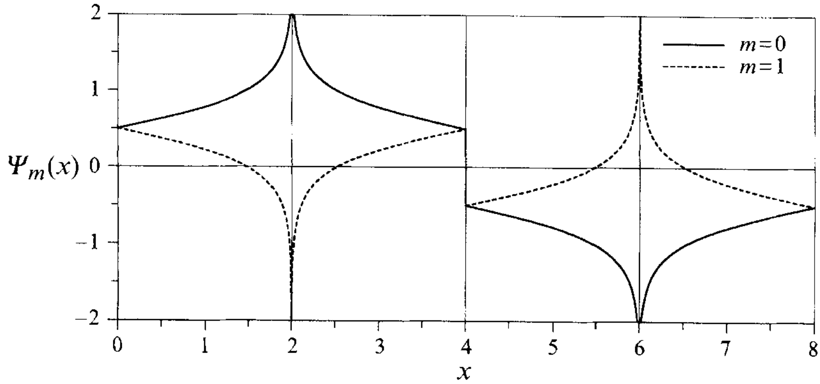

FIGURE 1. The functions $\Psi_0(x)$ and $\Psi_1(x)$ for $0 \leqslant x \leqslant 8$.

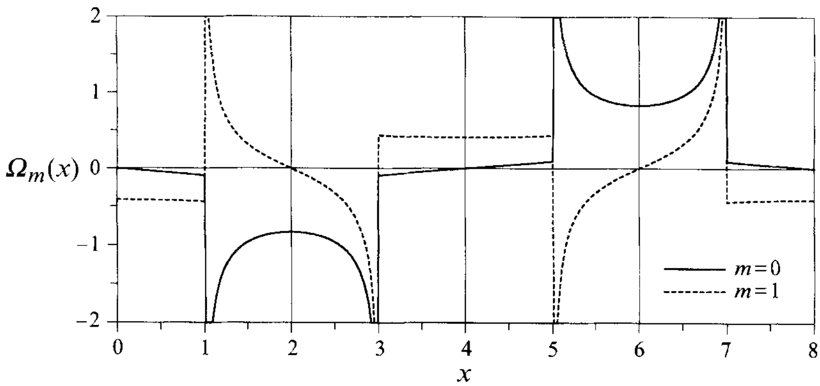

FIGURE 2. The functions $\Omega_0(x)$ and $\Omega_1(x)$ for $0 \leqslant x \leqslant 8$.

where $Z(x)$ denotes a function of period 1, which in $[0, 1]$ is identical with the special case $s = \frac{1}{2}$ of Riemann's generalized zeta function $\zeta(s, x)$ (cf. Whittaker & Watson 1927). Tables and computational relations for $\zeta(\frac{1}{2}, x)$ have been given by Powell (1952). Since for $x \to 0$, $\zeta(\frac{1}{2}, x)$ approaches infinity as $x^{-1/2}$, $Z(x)$ exhibits the same behaviour whenever its argument is an integer number. Consequently, the functions $\Omega_0(x)$ and $\Omega_1(x)$ are functions of period 8 and exhibit inverse-square-root singularities at $x = \pm 1, \pm 3, \pm 5 \ldots$ as shown in figure 2.

Since now both the comparison terms (3.25) as well as the closed comparison sums (3.26) are completely determined, Kummer's series transformation (3.18) can be applied to evaluate the derivatives (3.19) of $S_m(r, \zeta)$ and to discuss the nature and location of their singularities. On considering the arguments of $\Psi_m(x)$ in (3.26), we obtain the result that for $r > 0$, both $S_{m,r}(r, \zeta)$ and $S_{m,\zeta}(r, \zeta)$ have discontinuities at $(r \pm \zeta - 1) = 4k$ and logarithmic poles at $(r \pm \zeta - 1) = 4k + 2$, k being an integer number. If these locations are plotted in the (r, ζ)-plane, we obtain a periodic zig-zag Mach line pattern as illustrated in figure 3. On crossing the axis $r = 0$, the type of singularity is always changed from discontinuity to logarithmic pole or vice versa, while the type remains the same on reflection at the wall $r = 1$. In all cases $m > 1$, the axis remains undisturbed, whereas for $m = 0$ and $m = 1$, $S_{m,\zeta}(0, \zeta)$ and $S_{m,r}(0, \zeta)$ exhibit inverse-square-root singularities at $\zeta = 1, 3, 5 \ldots$, respectively, i.e. at the locations where the singularities cross the axis and change their type. Since

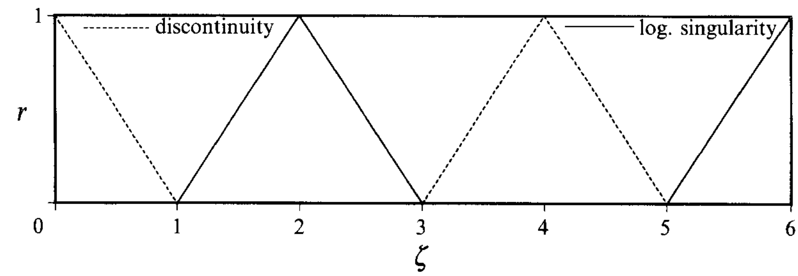

FIGURE 3. Location of singularities in the (r, ζ)-plane.

according to (3.13), the discussed singularities are present whenever abrupt bending of the boundary streamlimes occurs at $\zeta = 0$ and thus, in physical reality, shock or expansion waves are produced, the obtained results strongly corroborate the intuitive conjecture that the mathematical singularities represent physical phenomena which linear potential theory fails to describe correctly.

Besides discussing the singularities of the derivatives of $S_m(r, \zeta)$, we are now also able to perform their proper numerical evaluation by applying Kummer's series transformation (3.18). The practical need for this method and its efficiency are demonstrated in the following for the series

$$S_{0,\zeta}(0, \zeta) = -\sum_{n=2}^{\infty} \frac{2 \sin \beta'_{0n}\zeta}{\beta'_{0n} J_0(\beta'_{0n})}, \tag{3.30}$$

which describes the dimensionless perturbation of axial velocity on the axis for the axisymmetric case. Figure 4(*a*) shows the result of a direct evaluation of (3.30) *without* performing Kummer's series transformation. The non-uniform convergence produced by the singularities of $S_{0,\zeta}(0, \zeta)$ manifests itself in strong oscillations of the curve (Gibbs' phenomenon), which do not vanish no matter how many terms are included into the sum (30 terms in figure 4*a*). If Kummer's series transformation is now applied to $S_{0,\zeta}(0, \zeta)$, we obtain with the notations (3.19*b*), (3.25*b*) and (3.26*b*)

$$\begin{aligned} S_{0,\zeta}(0, \zeta) &= \sum_{n=1}^{\infty} s_{0n,\zeta}(0, \zeta) \\ &= \tilde{S}_{0,\zeta}(0, \zeta) + \sum_{n=1}^{\infty} \left[s_{0n,\zeta}(0, \zeta) - \tilde{s}_{0n,\zeta}(0, \zeta) \right] \\ &= \tilde{S}_{0,\zeta}(0, \zeta) + R_{0,\zeta}(0, \zeta), \end{aligned} \tag{3.31}$$

where

$$\tilde{S}_{0,\zeta}(0, \zeta) = \frac{1}{2} \left[Z\left(\frac{3+\zeta}{8}\right) - Z\left(\frac{3-\zeta}{8}\right) - Z\left(\frac{7+\zeta}{8}\right) + Z\left(\frac{7-\zeta}{8}\right) \right], \tag{3.32a}$$

$$R_{0,\zeta}(0, \zeta) = 2\sqrt{2} \sin \frac{\pi\zeta}{4} - \sum_{n=2}^{\infty} \left[\frac{2 \sin \beta'_{0n}\zeta}{\beta'_{0n} J_0(\beta'_{0n})} + (2\pi)^{1/2} (-1)^n \frac{\sin \mu'_{0n}\zeta}{(\mu'_{0n})^{1/2}} \right], \tag{3.32b}$$

and thus the singularities of $S_{0,\zeta}(0, \zeta)$ are shifted to the closed expression (3.32*a*), whereas the remainder series (3.32*b*) converges uniformly and thus approaches a continuous function after inclusion of 20–30 terms. Both the comparison sum

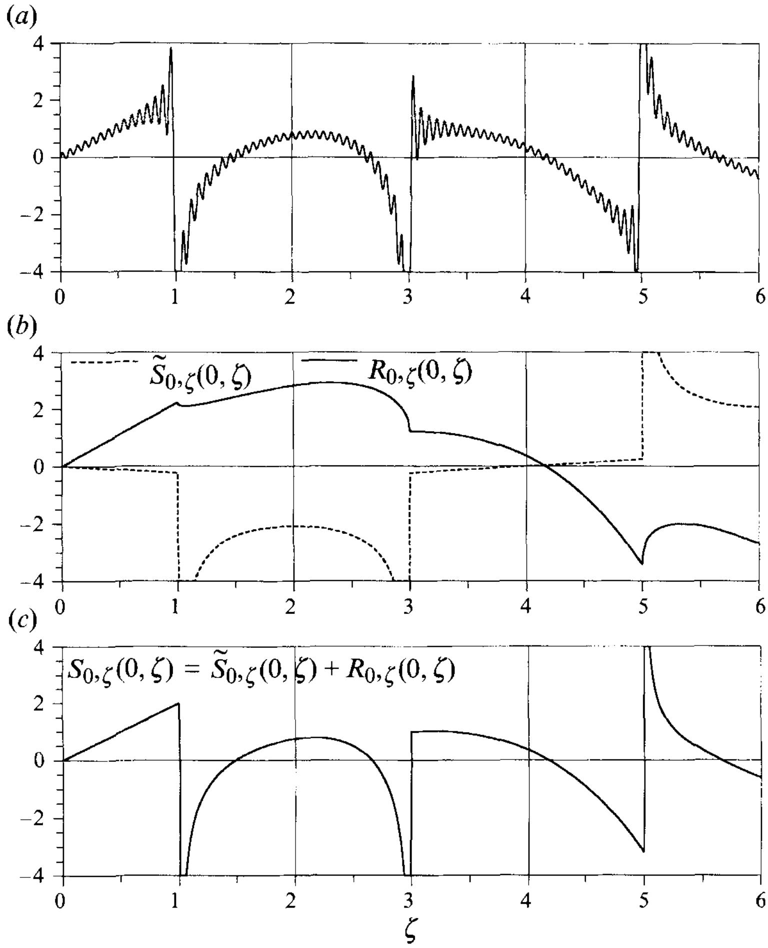

FIGURE 4. The efficiency of Kummer's series transformation demonstrated for the series $S_{0,\zeta}(0,\zeta)$. (*a*) Result of a direct computation using 30 terms of (3.30). (*b*) Closed comparison sum (3.32*a*) and continuous remainder series (3.32*b*) (30 terms). (*c*) $S_{0,\zeta}(0,\zeta)$ obtained by adding comparison sum and remainder series according to (3.31).

$\tilde{S}_{0,\zeta}(0,\zeta)$ and the continuous function $R_{0,\zeta}(0,\zeta)$ are presented in figure 4(*b*). Finally, figure 4(*c*) shows $S_{0,\zeta}(0,\zeta)$ obtained via (3.31), i.e. by simply adding $\tilde{S}_{0,\zeta}(0,\zeta)$ and $R_{0,\zeta}(0,\zeta)$. By comparing the result obtained by direct summation with the result provided by Kummer's series transformation, the advantage of this method becomes immediately obvious.

Consequently, by means of the results obtained in this section, we can immediately evaluate any solution (2.16) by using the following procedure.

(i) Decompose the function $\phi_m(r,\zeta)$ via (3.13) into a part $\Delta\lambda' S_m(r,\zeta)$ containing the singularities and a uniformly converging sum with uniformly converging first derivatives. (Note that this decomposition becomes trivial if $\Delta\lambda' = 0$, i.e. if no abrupt bending of the boundary streamlines occurs at $\zeta = 0$.)

(ii) Evaluate the uniformly converging part of the solution by direct summation and evaluate the derivatives of $S_m(r,\zeta)$ by Kummer's series transformation as described in this section. Because of the universal character of $S_m(r,\zeta)$, the results of this evaluation

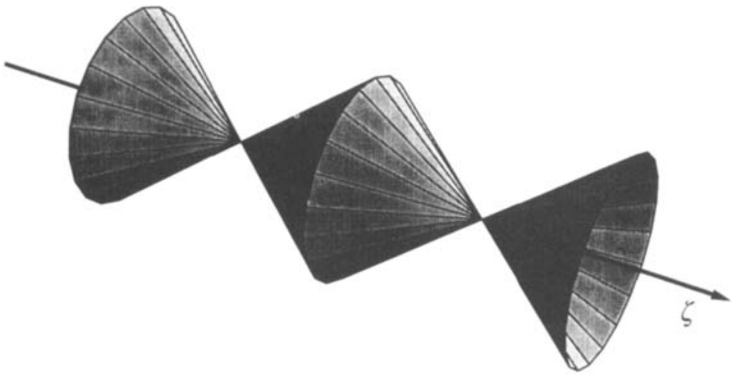

FIGURE 5. Characteristic surface consisting of alternating coaxial Mach cones.

need be tabulated just once, or the above evaluation scheme can be implemented as a function in a computer program.

4. Asymptotic reflection and transmission laws

Before now applying linear potential theory as derived in the previous sections, we will consider the general rules which govern the propagation of small perturbations in supersonic duct flow. It is well established in mathematical physics (cf. Courant & Hilbert 1931) that in the case of phenomena being described by hyperbolic partial differential equations like the wave equation (2.2), small disturbances propagate along characteristics, which thus represent 'natural coordinates' for the problem under study. As before, we will restrict ourselves to the special case (2.16) of harmonic azimuthal dependence of the velocity potential $\phi(r, \varphi, \zeta)$. If the solution ansatz (2.16) is substituted into the wave equation (2.2), we obtain the following partial differential equation to be satisfied by the function $\phi_m(r, \zeta)$:

$$\frac{\partial^2 \phi_m}{\partial r^2} + \frac{1}{r}\frac{\partial \phi_m}{\partial r} - \frac{m^2}{r^2}\phi_m - \frac{\partial^2 \phi_m}{\partial \zeta^2} = 0. \tag{4.1}$$

From the elementary theory of partial differential equations (cf. Courant & Hilbert 1931), it is easily verified that (4.1) is of hyperbolic type and that its characteristics are given by

$$r \pm \zeta = const. \tag{4.2}$$

which can be interpreted as an infinite family of Mach cones arranged coaxial to the flow axis $r = 0$ with their base circles and vertexes coinciding in an alternating manner as illustrated in figure 5. Furthermore, since (2.17) is the ordinary differential equation of a harmonic oscillator, it is clear that the coefficients $A_{mn}(\zeta)$ will essentially be given by harmonic functions; hence the velocity potential (2.16) will in general arise as an infinite linear combination of the spatial modes

$$\phi_{mn}(r, \varphi, \zeta) = J_m(\beta'_{mn} r)\, \mathrm{e}^{\mathrm{i}\beta'_{mn}\zeta} \cos m\varphi, \tag{4.3}$$

which by substituting the asymptotic expansions (3.21) for the Bessel function $J_m(x)$ and the zeros β'_{mn} of its derivatives, can be asymptotically decomposed into two parts propagating on Mach cones $r + \zeta = const.$ and $r - \zeta = const.$, respectively:

$$\phi_{mn}(r, \varphi, \zeta) \sim \Phi^+(r, \varphi, \zeta) + \Phi^-(r, \varphi, \zeta), \tag{4.4}$$

where

$$\Phi^{\pm}(r,\varphi,\zeta) = (-1)^{n+1}\left(\frac{2}{\pi\mu'_{mn}r}\right)^{1/2}\frac{1}{2}\,e^{\pm i\mu'_{mn}(r\pm\zeta-1)}\cos m\varphi .$$

Note that for $r \to 0$, the perturbations increase in magnitude as $r^{-1/2}$, which is physically plausible since on approaching the axis they are focussed into a single point. This characteristic feature of cylindrical supersonic flow has been mentioned by Ward (1948), who called the general phenomenon the 'radial focussing effect'. Mathematically, a perturbation (4.4) can thus be considered as being composed of two parts, each one travelling on an upstream and downstream characteristic, respectively. However, it is an experimentally verified fact that *physical* perturbations in supersonic flow can only travel on *downstream* characteristics. Thus, on following a perturbation on its way through the flow field, only its downstream component is to be considered. By means of the decomposition (4.4), we will in the following investigate the laws governing the asymptotic behaviour of its corresponding components for the case of transmission through the axis and reflection at the wall, respectively.

Consider a small harmonic perturbation which is originating, say, at the duct wall $r = 1$ at an arbitrary location $\zeta = \zeta^* - 1$, $\varphi = \varphi_0$ and is traveling towards the axis on the characteristic $r + \zeta = \zeta^*$. Hence, upon substituting the path of propagation into the corresponding part of (4.4), the perturbation can asymptotically be written as (C denotes a constant)

$$\Phi^{+}(r,\varphi,\zeta) = \frac{C}{r^{1/2}}\,e^{i\mu'_{mn}(\zeta^*-1)}\cos m\varphi_0 . \tag{4.5}$$

At $r = 0$, the perturbation crosses the axis, thus entering the half-plane $\varphi = \varphi_0 + \pi$ and changing to the characteristic $r - \zeta = -\zeta^*$. Consequently, we obtain under consideration of (3.21*c*)

$$\begin{aligned}\Phi^{-}(r,\varphi,\zeta) &= \frac{C}{r^{1/2}}\,e^{i\mu'_{mn}(\zeta^*+1)}\cos m(\varphi_0+\pi)\\ &= \frac{C}{r^{1/2}}\,e^{i\mu'_{mn}(\zeta^*-1)}\,e^{i\pi/2}\cos m\varphi_0\end{aligned} \tag{4.6}$$

and thus the basic result that upon crossing the axis, a perturbation suffers an asymptotical phase shift of $\pi/2$. In figure 6(*a*), this result is graphically illustrated in an axial section $\varphi_0, \varphi_0 + \pi$ for a disturbance which initially varies sinusoidally with respect to the surface of a Mach cone with vertex in $\zeta = \zeta_0^*$.

The case of a perturbation being reflected at the duct wall at $r = 1$, $\zeta = \zeta^*$ is treated in a quite similar manner. On travelling from the axis to the duct wall, the disturbance propagates along the downstream characteristic $r - \zeta = 1 - \zeta^*$. When reflected at the wall, it remains in the same half-plane $\varphi = \varphi_0$ and changes to the characteristic $r + \zeta = 1 + \zeta^*$. Hence, on substituting the propagation path into the corresponding components of (4.4), we obtain identical expressions:

$$\Phi^{+}(r,\varphi,\zeta) = \Phi^{-}(r,\varphi,\zeta) = \frac{C}{r^{1/2}}\,e^{i\mu'_{mn}\zeta^*}\cos m\varphi_0 \tag{4.7}$$

and thus the second basic result that asymptotically, the perturbation suffers no phase shift upon reflection at the wall of the duct, as illustrated in figure 6(*b*).

Because the first derivative of (4.3) with respect to ζ has the same functional form as (4.3) itself, the above results are valid for all flow variables (2.3) and (2.4) except for radial velocity, which depends on $\partial\phi/\partial r$. In this case, an expression similar to (4.4) can be derived on the basis of the asymptotic expansion (3.21*b*) for $J'_m(x)$ with

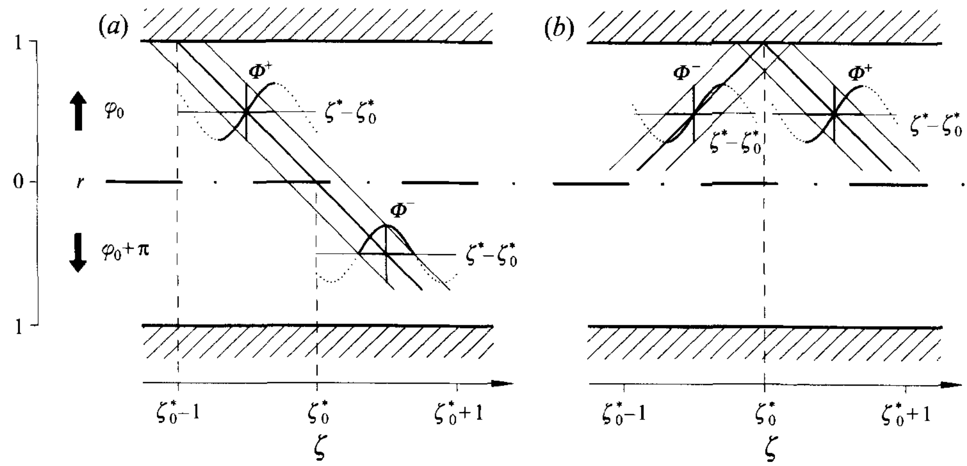

FIGURE 6. Asymptotic propagation laws for a sinusoidal perturbation. (*a*) Transmission through the axis $r = 0$ resulting in a phase shift of $\pi/2$; (*b*) reflection at the wall $r = 1$ without phase shift.

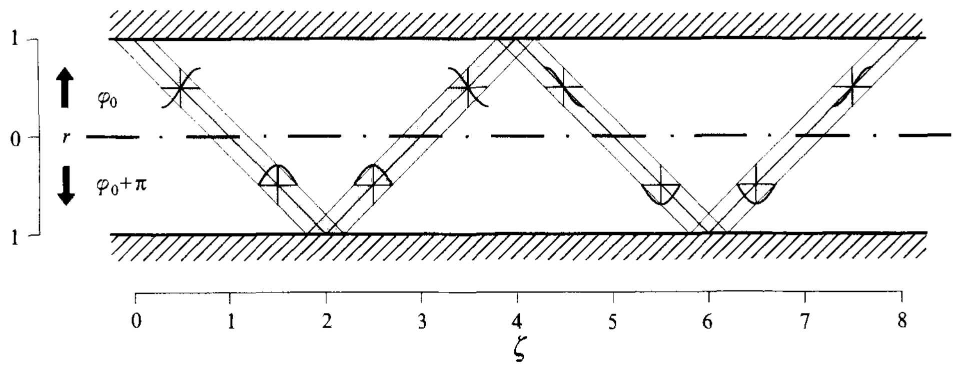

FIGURE 7. Asymptotic propagation laws for a sinusoidal density perturbation travelling through a cylindrical duct. Areas of compression are indicated by dark shading.

the result that a radial velocity perturbation suffers an asymptotical phase shift of $-\pi/2$ upon crossing the axis, and of π when reflected at the wall.

As an illustrative example, the propagation of an initially sinusoidal wave, e.g. a density perturbation, is presented in an axial section in figure 7. Starting at $r = 1$, $\varphi = \varphi_0$, $\zeta = 0$ with a skewsymmetric shape (i.e. expansive on the upstream side and compressive on the downstream side of the corresponding Mach cone), the perturbation is travelling towards the axis. On crossing the axis at $r = 0$, $\zeta = 1$ and entering the half-plane $\varphi = \varphi_0 + \pi$, the perturbation suffers a phase shift of $\pi/2$, being thereby transformed into a symmetric compression wave, i.e. its symmetry with respect to the characteristic is changed. The transformed wave is reflected without change at $r = 1$, $\varphi = \varphi_0 + \pi$, $\zeta = 2$ back towards the axis. The second crossing of the axis at $r = 0$, $\zeta = 3$, results again in a phase shift of $\pi/2$, i.e. in the transformation into a skewsymmetric compression–expansion wave, which is propagating back into the half-plane φ_0. At $r = 1$, $\zeta = 4$, it is again reflected towards the axis $r = 0$, $\zeta = 5$, where it is transformed into a symmetric expansion wave when entering the half-plane $\varphi = \varphi_0 + \pi$. Finally, after identical reflection at $r = 1$, $\zeta = 6$, the perturbation is

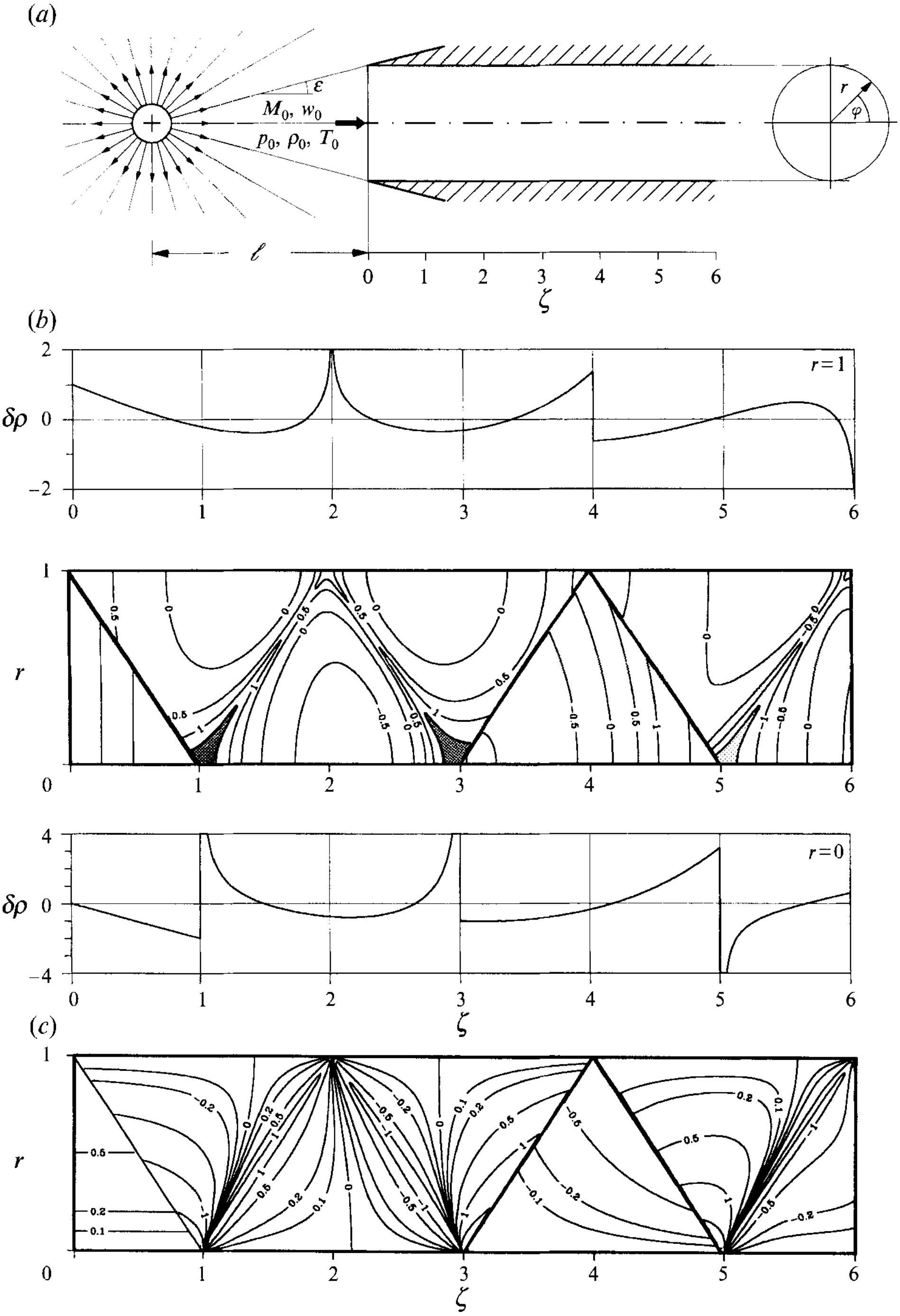

FIGURE 8. Spherical inflow into an ideal cylinder. (*a*) Problem geometry; (*b*) dimensionless density perturbation $\delta\rho(r,\zeta)$ at the duct wall (top), inside the duct (middle) and on the duct axis (bottom); (*c*) dimensionless perturbation $\delta u(r,\zeta)$ of radial velocity.

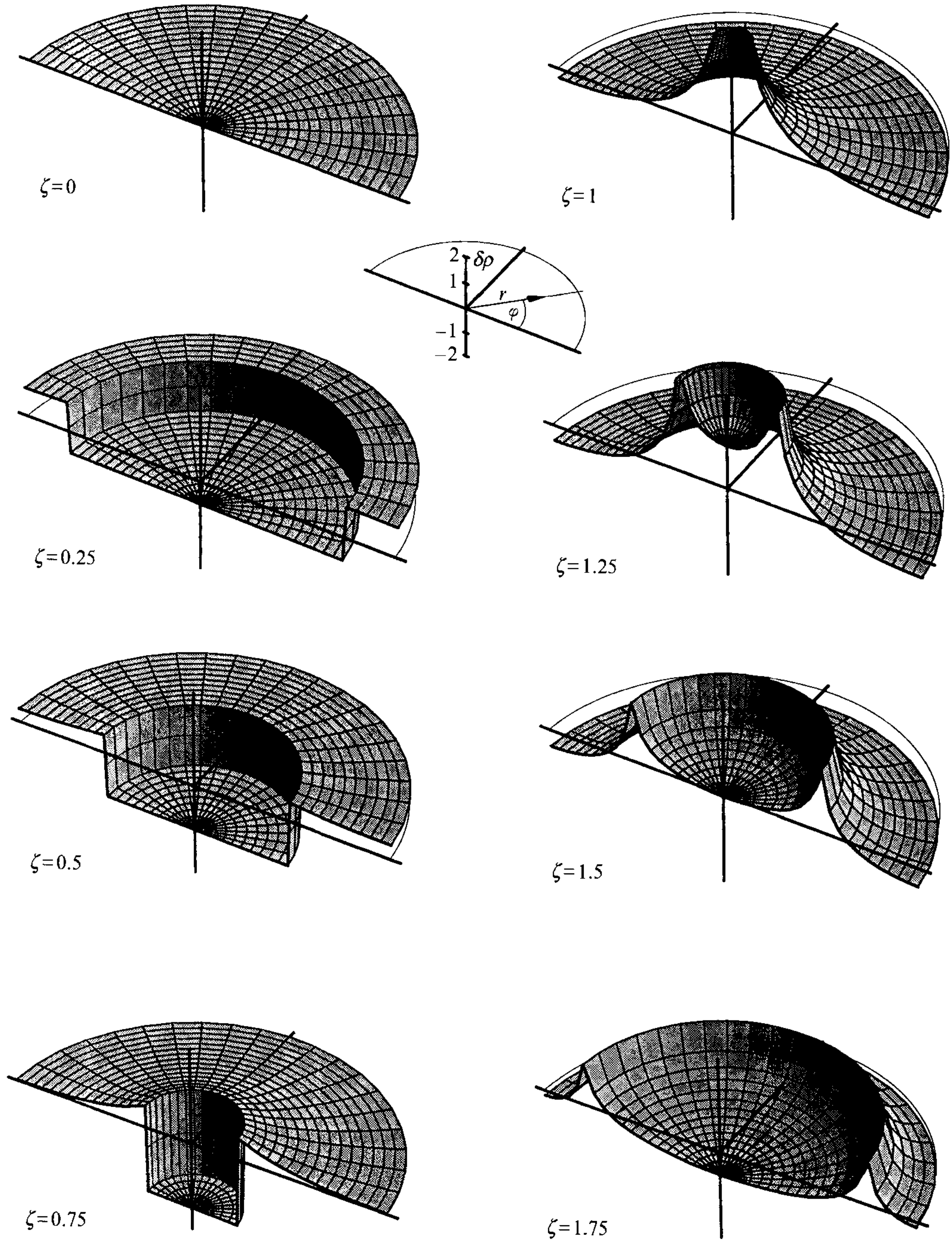

FIGURE 9. Spherical inflow into an ideal cylinder. Surface plots of the dimensionless density perturbation $\delta\rho(r,\zeta)$ in the semicircle $0 \leqslant r \leqslant 1$, $0 \leqslant \varphi \leqslant \pi$ at various locations $\zeta = const.$

transformed back into its initial expansion–compression shape upon crossing the axis at $\zeta = 7$, so that after reflection at $r = 1$, $\zeta = 8$ the whole cycle starts again.

If now a flow variable is mathematically represented by an infinite non-uniformly converging series in terms of the elementary modes (4.3), then its singular behaviour is solely determined by the large-n asymptotic form of these modes (cf. § 3). Since the variation in the immediate neighbourhood of a characteristic is the same for all modes, it is evident that in the case of non-uniform convergence, a sinusoidal variation gives rise to a discontinuity, while the symmetric cosine variation results in a logarithmic pole. Thus, the singularity pattern in figure 3 is easily explained. In addition, since the single modes vary asymptotically in magnitude as $r^{-1/2}$, regions of extreme magnitudes of the flow variable are to be expected in the vicinity of a logarithmic singularity close to the axis. These qualitative conclusions will be confirmed in the following section.

Thus, the basic mechanisms of supersonic flow in cylindrical ducts have been qualitatively identified for the case of harmonic azimuthal dependence of the velocity potential. On approaching the axis, the perturbations increase in magnitude asymptotically as $r^{-1/2}$, while they decrease in the same manner as the wall is approached. To a first degree of approximation, all perturbations are reflected without phase shift except the velocity component normal to the wall, which is reflected with a phase shift of π. Note that for plane flow, the same results are exactly valid. A very interesting phenomenon is the phase shift of $\pm\pi/2$ upon transmission through the axis, which changes the symmetry properties of a wave with respect to the characteristic and has no analogy in plane flow.

5. Examples of supersonic flow in ducts

5.1. *Spherical inflow into an ideal cylinder*

Consider an ideal right cylinder of constant circular cross-section with its entry at $\zeta = 0$. A spherical sonic source with purely radial supersonic flow field is located at $\zeta = -\ell$ in front of the cylinder on its axis, as illustrated in figure 8(*a*). (Note that such a sonic source represents an exact solution of the nonlinear equations of motion, cf. Oswatitsch 1952.) If we assume the distance ℓ to be large compared to unity, i.e. to the radius of the cylinder, then we can expand the spherical velocity field in a Taylor series in the entry plane $\zeta = 0$ to obtain the approximate velocity components u, v, w in cylindrical polar coordinates (r, φ, ζ):

$$\frac{u}{w_0} = \frac{\varepsilon}{(M_0^2-1)^{1/2}} r + O(\varepsilon^3), \quad \frac{v}{w_0} = 0, \quad \frac{w}{w_0} = 1 + O(\varepsilon^2), \tag{5.1a,b,c}$$

where

$$\varepsilon = \frac{1}{\ell} \ll 1 \tag{5.1d}$$

and the normalization velocity w_0 corresponds to the velocity of the source flow field at radial distance ℓ from the centre of the source. (The same holds for all normalization quantities M_0, p_0, ρ_0 and T_0.) Consequently, upon neglecting squares and higher powers of the geometry parameter ε, we obtain via (2.3) the following initial conditions (2.7):

$$\phi\big|_{\zeta=0} = \frac{\varepsilon}{(M_0^2-1)^{1/2}} \left(\frac{r^2}{2} + \text{const.}\right), \tag{5.2a}$$

$$\frac{\partial\phi}{\partial\zeta}\bigg|_{\zeta=0} = 0, \tag{5.2b}$$

while for the cylinder, the boundary condition (2.6) is simply given by

$$\lambda(\varphi,\zeta) = 0, \qquad \zeta > 0. \tag{5.2c}$$

Hence, from (2.19) and (2.20), we obtain the velocity potential

$$\phi(r,\zeta) = \frac{\varepsilon}{(M_0^2-1)^{1/2}} \sum_{n=2}^{\infty} \frac{2\,J_0(\beta'_{0n})\cos\beta'_{0n}\zeta}{\beta'^2_{0n}\,J_0(\beta'_{0n})} = \frac{\varepsilon}{(M_0^2-1)^{1/2}}\, S_0(r,\zeta), \tag{5.3}$$

where the free constant in (5.2a) has been chosen as $-1/4$ in order to avoid constant terms in (5.3), cf. (3.9a). Thus, the physical meaning of the series $S_m(r,\zeta)$ is obvious for the axisymmetric case $m=0$.

By means of (2.3) and (2.4), all physical flow variables of interest can now be determined in a straightforward manner. If we introduce the dimensionless perturbations $\delta\rho(r,\zeta)$ and $\delta u(r,\zeta)$, we obtain for example

$$\frac{\rho}{\rho_0} = 1 + \varepsilon\frac{M_0^2}{M_0^2-1}\,\delta\rho(r,\zeta), \tag{5.4a}$$

$$\frac{u}{w_0} = \frac{\varepsilon}{(M_0^2-1)^{1/2}}\,\delta u(r,\zeta), \tag{5.4b}$$

where

$$\delta\rho(r,\zeta) = -S_{0,\zeta}(r,\zeta), \quad \delta u(r,\zeta) = S_{0,r}(r,\zeta). \tag{5.5a, b}$$

Contour plots of the dimensionless perturbations (5.5) are presented in figures 8(b) and 8(c), respectively. In order to provide an impression of the radial density profiles, figure 9 additionally shows surface plots of $\delta\rho(r,\zeta)$ in the semi-circle $0\leqslant r\leqslant 1$, $0\leqslant\varphi\leqslant\pi$ for various locations $\zeta=$ const. At the entry $\zeta=0$ of the tube, all boundary streamlines are bent abruptly towards the axis, thus inducing compression waves propagating along the downstream surface of the leading Mach cone with base circle in $\zeta=0$ and vertex in $\zeta=1$, while the flow inside the cone is further expanding. Thus, on the leading characteristic $r+\zeta=1$, a discontinuous perturbation which is expansive on the upstream side and compressive on the downstream side of the characteristic is propagating towards the axis. As already mentioned at the end of § 4, this situation corresponds qualitatively to the example discussed in figure 7. Therefore, the spatial structure of the flow field can be explained by the asymptotic transmission and reflection laws of the previous section, keeping in mind that the non-uniform convergence of the derivatives of $S_0(r,\zeta)$ will produce logarithmic poles or discontinuities at those locations where the asymptotic perturbations (4.4) exhibit symmetry or skewsymmetry, respectively.

By radial focussing, the compressive parts of the discontinuous wave increase strongly in magnitude on approaching the vertex of the leading Mach cone. On transmission through the axis, the discontinuous expansion–compression wave is transformed into a wave which is purely compressive on both sides of the leading characteristic and produces a region of extremely high density (indicated by dark shading in figure 8b) in the downstream vicinity of $r=0$, $\zeta=1$. At the wall of the tube at $r=1$, $\zeta=2$, the purely compressive wave is reflected as a wave of the same type in agreement with the asymptotic reflection law and, by radial focussing, produces again a high-density region on the axis in the upstream vicinity of $\zeta=3$. On once more crossing the axis, the purely compressive wave is transformed back into a discontinuous wave, which now is compressive on the upstream side and expansive on the downstream side of the characteristic $r-\zeta=3$. Being reflected at the wall at $\zeta=4$, the compressive–expansive perturbation travels back to the axis, where at $\zeta=5$, it is again transformed into a purely expansive perturbation forming a region of extremely low density (indicated by lighter shading in figure 8*b*) in the upstream vicinity of $r = 0$, $\zeta = 5$. Finally, upon reflection at the wall at $\zeta = 6$, an identical low-density region is formed in the upstream vicinity of $r = 0$, $\zeta = 7$; and after transmission through the axis and reflection of the resulting discontinuous expansion–compression wave at $r = 1$, $\zeta = 8$, the whole cycle starts again. This process is continuously repeated for increasing values of ζ; because of the incommensurable eigenvalues β'_{0n} however, the wave pattern is never strictly periodic and thus much more complicated compared to plane flow, where the wave type remains unchanged and the flow field is strictly periodic without singularities.

Obviously, the cylindrical geometry causes a complex flow field, whose main features are non-periodicity, radial focussing and the transformation of wave type by crossing of the axis. Since according to (3.13), the series $S_0(r,\zeta)$ occurs whenever a discontinuity of wall slope is present at $\zeta = 0$, the qualitative validity of the above discussion is of universal character and not restricted to the simple case considered here.

5.2. *Coaxial inflow into an axisymmetric pipe with linearly varying cross-section*

A problem closely related to the example discussed above is the uniform coaxial inflow into a pipe whose cross-section varies linearly with the axial coordinate. The formal solution of this problem and a discussion of the singularities of the flow field was first given by Ward (1948). However, Ward did not evaluate his formal solution because of the non-uniform convergence of the Dini series involved.

Consider a uniform coaxial supersonic flow with velocity w_0 and Mach number M_0 entering a pipe with linearly varying cross-section as sketched in figure 10(*a*). The problem is described by the following initial and boundary conditions:

$$\phi\big|_{\zeta=0} = \frac{\partial\phi}{\partial\zeta}\bigg|_{\zeta=0} = 0, \tag{5.6a}$$

$$\lambda(\varphi,\zeta) = \varepsilon\,\zeta, \tag{5.6b}$$

where the geometry parameter ε denotes the constant slope of the contour function, which for small positive or negative values is approximately equal to the angle of divergence or convergence of the duct contour, respectively. By substituting (5.6) into (2.19), (2.20), we obtain

$$A_{0n}(\zeta) = \frac{\varepsilon}{(M_0^2-1)^{1/2}}\,\frac{2\left[1-\cos\beta'_{0n}\zeta\right]}{\beta'^2_{0n}\,J_0(\beta'_{0n})}. \tag{5.7}$$

Since $\beta'_{01} = 0$, an undefined expression of the form '0/0' arises for $n = 1$; consequently, $A_{01}(\zeta)$ must be determined by a limiting process. By applying de l'Hospital's rule, we obtain from (5.7)

$$A_{01}(\zeta) = \frac{\varepsilon}{(M_0^2-1)^{1/2}}\,\zeta^2. \tag{5.8}$$

Hence, the velocity potential assumes the following form:

$$\phi(r,\zeta) = \frac{\varepsilon}{(M_0^2-1)^{1/2}}\left[\zeta^2 + \sum_{n=2}^{\infty}\frac{2\,J_0(\beta'_{0n})\left[1-\cos\beta'_{0n}\zeta\right]}{\beta'^2_{0n}\,J_0(\beta'_{0n})}\right] \tag{5.9}$$

where the infinite series in brackets converges non-uniformly in consequence of the singularities caused by abrupt bending of the boundary streamlines at $\zeta = 0$. By applying the decomposition (3.13), we can separate the singular part in terms of

$S_0(r,\zeta)$, while the remainder series is identical to (3.9*a*) and can thus be written in closed form:

$$\phi(r,\zeta) = \frac{\varepsilon}{(M_0^2-1)^{1/2}} \left[\zeta^2 + \tfrac{1}{2}(r^2 - \tfrac{1}{2}) - S_0(r,\zeta)\right]. \tag{5.10}$$

Hence, the evaluation of the flow field has again essentially been reduced to the evaluation of the derivatives of $S_0(r,\zeta)$. In a similar manner as in the previous example, we can for example define the dimensionless density perturbation $\delta\rho(r,\zeta)$:

$$\frac{\rho}{\rho_0} = 1 + \varepsilon \frac{M_0^2}{M_0^2 - 1} \delta\rho(r,\zeta), \tag{5.11}$$

where

$$\delta\rho(r,\zeta) = S_{0,\zeta}(r,\zeta) - 2\zeta \tag{5.12}$$

and ρ_0 denotes the density in the undisturbed coaxial inflow.

If for example we consider a diverging tube (i.e. $\varepsilon > 0$), the flow can be regarded as a superposition of a pure expansion due to the increase of cross-section (described by the first two terms in the square brackets of (5.10)) and, because of the *negative* sign of $S_0(r,\zeta)$, a *converging* inflow into an undeformed tube. In the interior of the leading Mach cone, both components cancel out and thus the flow remains undisturbed, while in consequence of the outward bending of the streamlines at the tube edge $r = 1$, $\zeta = 0$, an expansion wave is propagating towards the axis to end up as a region of extremely low density in the downstram vicinity of $r = 0$, $\zeta = 1$. The dimensionless density perturbation is presented in figure 10(*b*). It is obvious from (5.10) that besides the superimposed pure expansion effect, the whole discussion of the flow field can be done analogously to the previous example by simply exchanging compression for expansion and vice versa. Since however the magnitude of the density perturbation is increasing linearly without limit for increasing ζ, it is clear that even for small values of the divergence rate ε, the validity of the solution (5.10) is limited to values of ζ which are much smaller than $1/\varepsilon$.

5.3. *Coaxial inflow into a wavy axisymmetric duct*

While the supersonic flow past a wavy wall represents a well-known textbook example for the case of plane flows, no solution has been found yet for the case of internal flow with cylindrical geometry. For the axisymmetric case of uniform coaxial inflow into a sinusoidally deformed tube starting with an initial contour contraction, the initial and boundary conditions can be written as follows:

$$\phi\big|_{\zeta=0} = \frac{\partial\phi}{\partial\zeta}\bigg|_{\zeta=0} = 0, \tag{5.13a}$$

$$\lambda(\varphi,\zeta) = -\varepsilon \sin\omega\zeta, \tag{5.13b}$$

with ε being the dimensionless amplitude of the contour function and ω its spatial frequency with respect to the dimensionless spatial coordinate ζ. By substituting (5.13) into (2.19) and (2.20) and using the identity

$$\sum_{n=1}^{\infty} \frac{2\,J_0(\beta'_{0n}r)}{(\beta'^2_{0n}-\omega^2)\,J_0(\beta'_{0n})} = \frac{J_0(\omega r)}{\omega\,J'_0(\omega)}, \tag{5.14}$$

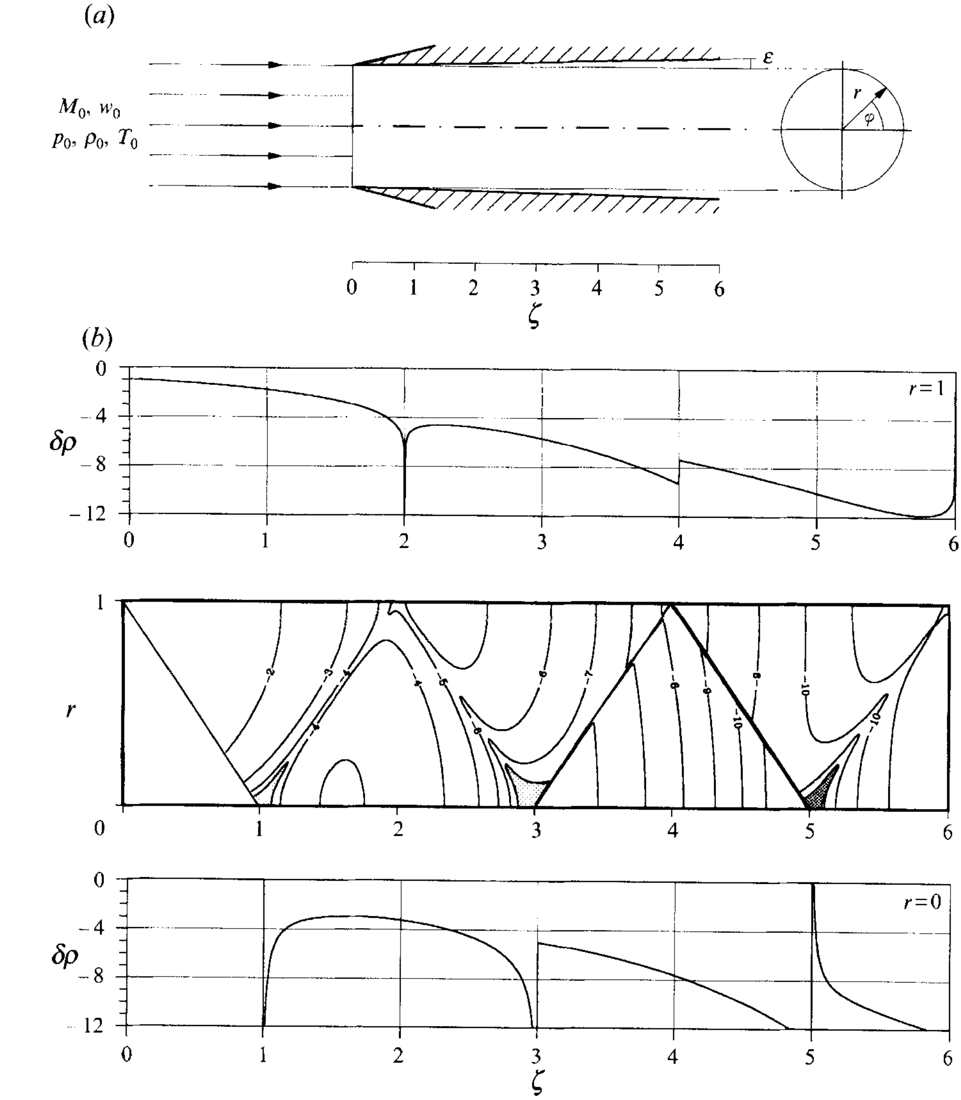

FIGURE 10. Coaxial inflow into a linearly diverging axisymmetric pipe. (*a*) Problem geometry; (*b*) dimensionless density perturbation $\delta\rho(r,\zeta)$ at the duct wall (top), inside the duct (middle) and on the duct axis (bottom).

which can be proved by expanding the function $J_0(\omega r)$ into a Dini series via the integral relation (2.18), we obtain the velocity potential

$$\phi(r,\zeta) = \frac{\varepsilon\omega}{(M_0^2-1)^{1/2}}\left[\sum_{n=1}^{\infty}\frac{2\,J_0(\beta'_{0n}r)\,\cos\beta'_{0n}\zeta}{(\beta'^2_{0n}-\omega^2)\,J_0(\beta'_{0n})} - \frac{J_0(\omega r)}{\omega\,J'_0(\omega)}\cos\omega\zeta\right]$$

$$= \frac{\varepsilon\omega}{(M_0^2-1)^{1/2}}\left[R(r,\zeta) - \frac{J_0(\omega r)}{\omega\,J'_0(\omega)}\cos\omega\zeta\right], \qquad (5.15)$$

provided that ω does not coincide with one of the zeros β'_{0n} of $J'_0(x)$ (in this case, we obtain a resonance solution whose magnitude is increasing without limit for increasing

ζ). Similarly, if the duct geometry is defined by the contour function

$$\lambda(\varphi, \zeta) = \varepsilon \cos \omega\zeta \tag{5.16}$$

we obtain the velocity potential

$$\begin{aligned}\phi(r, \zeta) &= \frac{\varepsilon\omega}{(M_0^2 - 1)^{1/2}} \left[\omega \sum_{n=1}^{\infty} \frac{2 J_0(\beta'_{0n} r) \sin \beta'_{0n}\zeta}{\beta'_{0n} (\beta'^2_{0n} - \omega^2) J_0(\beta'_{0n})} - \frac{J_0(\omega r)}{\omega J'_0(\omega)} \sin \omega\zeta \right] \\ &= \frac{\varepsilon\omega}{(M_0^2 - 1)^{1/2}} \left[T(r, \zeta) - \frac{J_0(\omega r)}{\omega J'_0(\omega)} \sin \omega\zeta \right].\end{aligned} \tag{5.17}$$

Although the both solutions (5.15) and (5.17) look formally very similar at a first glance, there is a fundamental difference between them. While the series $T(r, \zeta)$ and its first derivatives converge uniformly and thus all flow variables described by (5.17) are bounded and continuous throughout the field, this is not the case for the first derivatives of the series $R(r, \zeta)$. Consequently, the flow field described by (5.15) exhibits the same singularities as the ones discussed in the previous subsections. From a physical point of view, the difference between the two solutions is due to the abrupt bending of the boundary streamlines in the sinusoidal case (5.15), whereas in the case of (5.17), the duct shape starts with zero initial slope and thus the initial boundary streamline deflection is relatively smooth. Consequently, depending on the initial slope of the wall contour, the flow patterns obtained in wavy ducts of the same deformation rate ε and spatial deformation frequency ω can be completely different.

For the numerical evaluation of the series expressions describing the flow variables in the case of sinusoidally deformed ducts, the singular part of the series $R(r, \zeta)$ must be written in terms of $S_0(r, \zeta)$. By applying the decomposition (3.13) to (5.15), we obtain

$$R(r, \zeta) = S_0(r, \zeta) + \omega^2 \sum_{n=2}^{\infty} \frac{2 J_0(\beta'_{0n} r) \cos \beta'_{0n}\zeta}{\beta'^2_{0n} (\beta'^2_{0n} - \omega^2) J_0(\beta'_{0n})} - \frac{2}{\omega^2}, \tag{5.18}$$

where the infinite series on the right-hand side and its first derivatives converge uniformly, thus representing continuous bounded functions throughout the field.

In figures 11 and 12, the dimensionless density perturbations $\delta\rho(r, \zeta)$ resulting from (5.15)

$$\delta\rho(r, \zeta) = -\frac{J_0(\omega r)}{J'_0(\omega)} \sin \omega\zeta \; - \frac{\partial R(r, \zeta)}{\partial \zeta} \tag{5.19}$$

and (5.17)

$$\delta\rho(r, \zeta) = \frac{J_0(\omega r)}{J'_0(\omega)} \cos \omega\zeta \; - \frac{\partial T(r, \zeta)}{\partial \zeta} \tag{5.20}$$

are presented for the spatial deformation frequencies $\omega = \pi/3$ and $2\pi/3$. From the graphical representation, the fundamental differences between the two cases are clearly obvious. While in the case of sinusoidal deformation, the density pattern is mainly determined by the singularities and radial focussing and thus looks similar to the cases discussed above, the velocity potential (5.17) describes a continuous density field where singularities occur only in the density *gradients*, thus leading to kinks in the lines of constant density (indicated by the thin zigzag lines in figure 12).

Note that for the case of the contour function $\lambda(\zeta)$ being a general periodic function with period ω, the velocity potential $\phi(r, \zeta)$ appears as an (infinite) linear combination of potentials of the type (5.15) or (5.17) with *commensurable* spatial frequencies $n\omega$, $n = 0, 1, 2, \dots$. It is interesting that in this case, a part of the solution arises in the form of so-called Schlömilch series, i.e. series of the form

$$\sum_{n=1}^{\infty} c_n J_0(n\omega r)$$

(cf. Watson 1944), which thus present themselves naturally as particular solutions in the problem of supersonic flow in periodically deformed cylindrical ducts.

5.4. *Parallel flow into an inclined ideal cylinder*

In order to complete our discussion of elementary supersonic flows in cylindrical ducts, we will finally consider a basic case of non-axisymmetric flow. Consider an undeformed ideal cylinder at small incidence angle ε (measured in the (r, φ, ζ)-system) to a uniform parallel flow with velocity w_0, Mach number M_0, density ρ_0 etc. as illustrated in figure 13. The initial and boundary conditions of the problem are

$$\phi\big|_{\zeta=0} = \frac{\varepsilon}{(M_0^2 - 1)^{1/2}}\, r \cos\varphi, \tag{5.21a}$$

$$\frac{\partial\phi}{\partial\zeta}\bigg|_{\zeta=0} = 0, \tag{5.21b}$$

$$\lambda(\varphi, \zeta) = 0. \tag{5.21c}$$

By substituting (5.21) into (2.19) and (2.20), we obtain the following velocity potential:

$$\begin{aligned}\phi(r, \varphi, \zeta) &= \frac{\varepsilon}{(M_0^2 - 1)^{1/2}} \cos\varphi \sum_{n=1}^{\infty} \frac{2 J_1(\beta'_{1n} r) \cos\beta'_{1n}\zeta}{(\beta'^2_{1n} - 1) J_1(\beta'_{1n})} \\ &= \frac{\varepsilon}{(M_0^2 - 1)^{1/2}} \cos\varphi\, S_1(r, \zeta).\end{aligned} \tag{5.22}$$

Thus, the flow under study can be seen as the most simple three-dimensional analogy to the axisymmetric problem treated in § 5.1, since it is completely determined by $S_1(r, \zeta)$. The density field is given by

$$\frac{\rho}{\rho_0} = 1 + \varepsilon \frac{M_0^2}{M_0^2 - 1}\, \delta\rho(r, \varphi, \zeta) \tag{5.23}$$

with $\delta\rho(r, \varphi, \zeta)$ being the dimesionless density perturbation

$$\delta\rho(r, \varphi, \zeta) = -S_{1,\zeta}(r, \zeta) \cos\varphi. \tag{5.24}$$

A contour plot of $\delta\rho(r, \varphi, \zeta)$ in the half-plane $\varphi = 0$ is given in figure 13. Since the corresponding three-dimensional flow field is somewhat hard to imagine from a contour plot, surface plots of $\delta\rho(r, \varphi, \zeta)$ at various locations $\zeta = const.$ are in addition presented in figure 14. Similarly to the axisymmetric case discussed in § 5.1, perturbations originate on the circumference of the duct edge at $r = 1$, $\zeta = 0$, but in the present case they are modulated by the cosine of the azimuthal angle φ. At $\varphi = 0$, the boundary streamlines are bent towards the axis and thus compression waves are propagating along the downstream surface of the leading Mach cone, while at $\varphi = \pi$, the streamlines are bend away from the axis resulting in an expansion wave of identical magnitude. At $\varphi = \pi/2$, the streamlines remain undistorted and, consequently, no

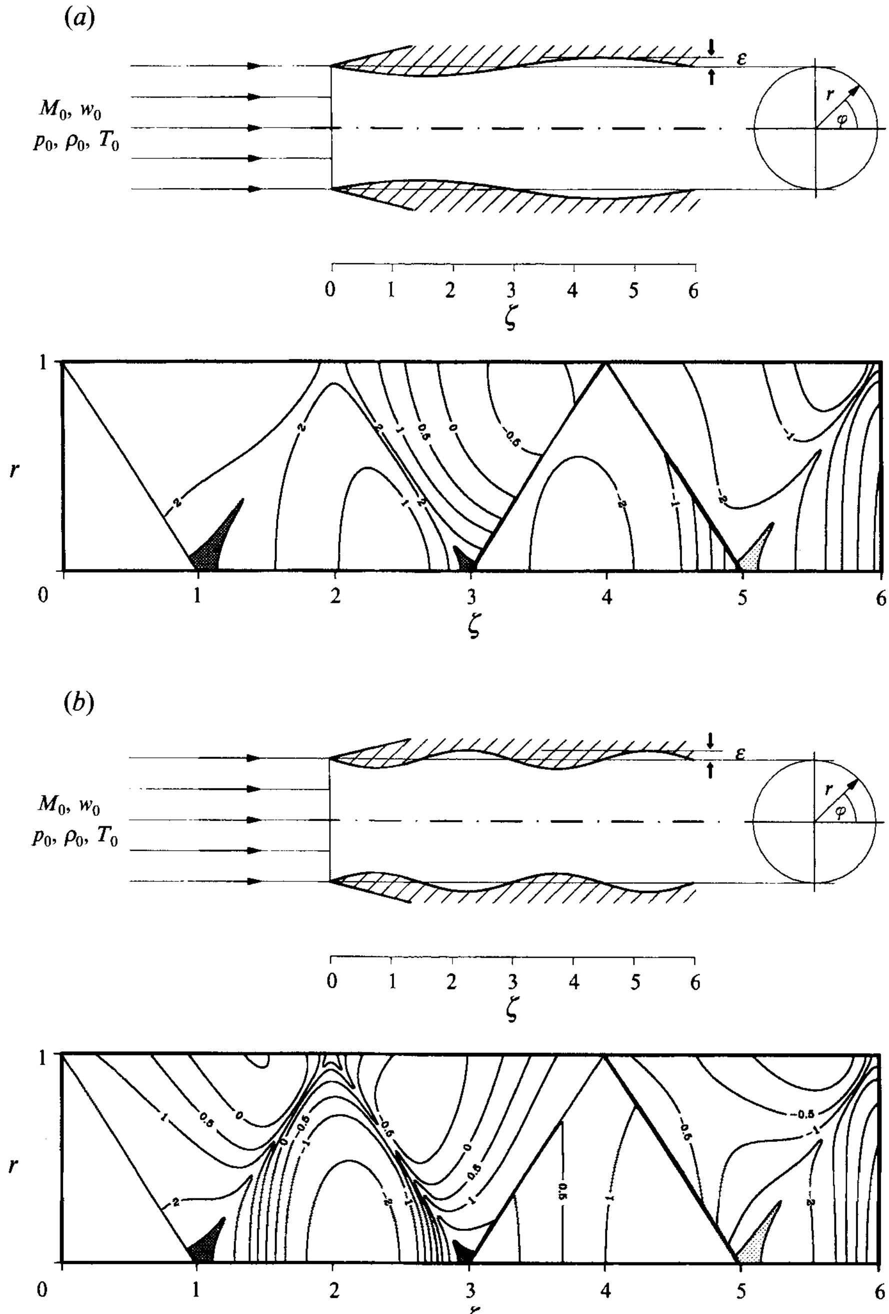

FIGURE 11. Coaxial inflow into a wavy axisymmetric pipe whose cross-section varies as $-\sin\omega\zeta$. (*a*) $\omega = \pi/3$; (*b*) $\omega = 2\pi/3$. Problem geometry (top) and $\delta\rho\ (r,\zeta)$ (bottom).

expansion or compression waves are produced. Close to the axis, radial focussing occurs as in the axisymmetric case to produce regions of very high and low density in the vicinity of $\zeta = 1$. On crossing the axis, the waves are transformed from the discontionuous skewsymmetric type into the logarithmic symmetric type to travel

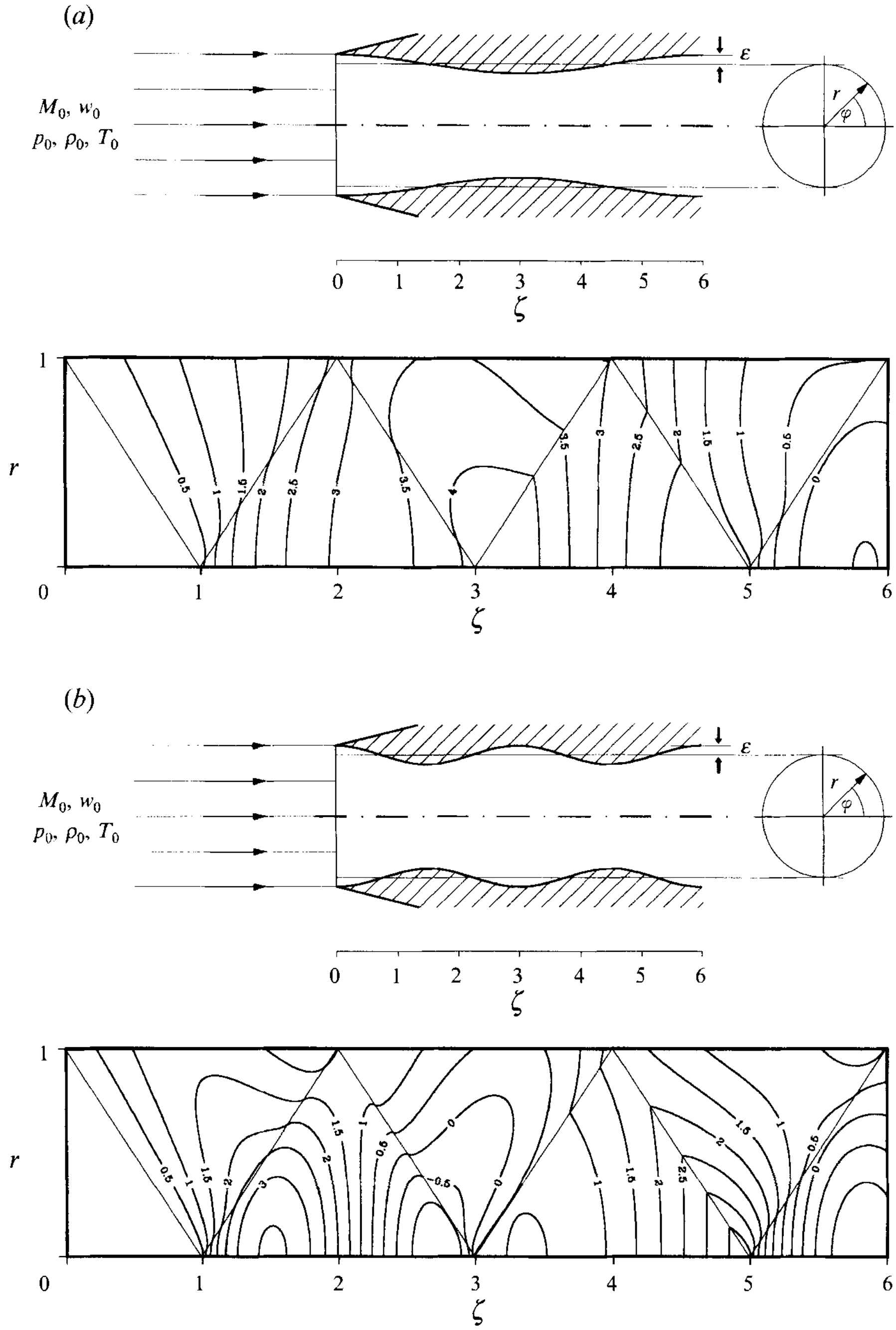

FIGURE 12. Coaxial inflow into a wavy axisymmetric pipe whose cross-section varies as $\cos \omega\zeta$. (*a*) $\omega = \pi/3$; (*b*) $\omega = 2\pi/3$. Problem geometry (top) and $\delta p\ (r, \zeta)$ (bottom).

further towards the wall. Thus, the whole physical mechanism works very similarly to the axisymmetric case, the only qualitative difference consisting in the asymmetry due to the modulation by $\cos \varphi$ and the fact that the axis remains undisturbed, since all perturbations cancel out there.

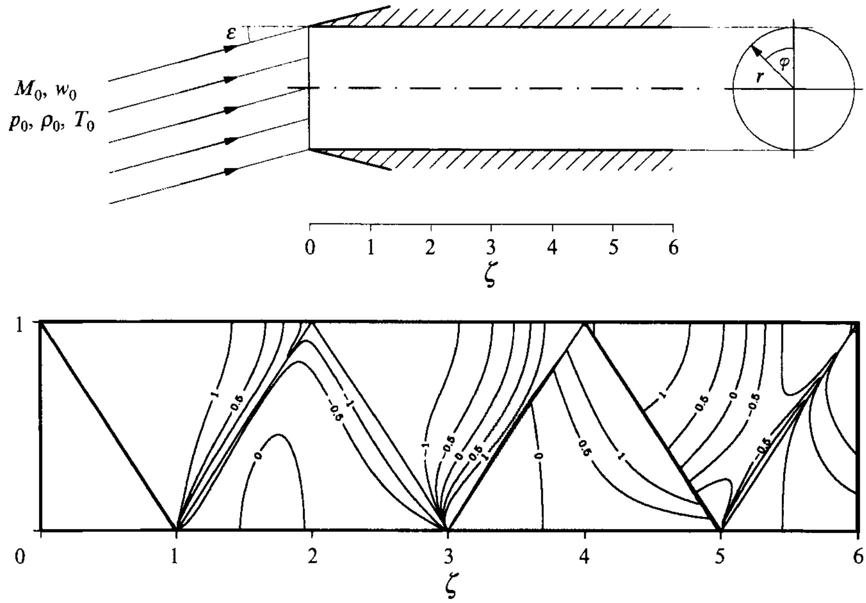

FIGURE 13. Parallel inflow into an inclined ideal cylinder. Problem geometry (top) and dimensionless density perturbation $\delta\rho(r, \varphi, \zeta)$ in the half-plane $\varphi = 0$ (bottom).

6. Summary and conclusions

In this paper, the linear potential theory of steady internal supersonic flow in slightly deformed quasi-cylindrical ducts has been presented. By decomposition of the flow into a uniform parallel flow and small irrotational perturbations, the mathematical formulation in § 2 leads to an initial-boundary value problem for the wave equation (2.2). While the inhomogeneous boundary condition (2.6) requires the radial velocity component at the wall to be proportional to axial duct slope, the initial conditions (2.7) describe the velocity perturbations at the entry of the duct. The general solution (2.9) arises as an infinite double series, which can be interpreted as a harmonic Fourier series in the azimuthal angle φ with each coefficient being itself a Dini series, whose coefficients obey the well-known ordinary differential equation (2.14) of a mass–spring-system with external exciting force. Thus, the problem of determining the complete flow inside a quasi-cylindrical duct with arbitrary three-dimensional contour function and inflow conditions can in general be considered as formally solved.

As was first pointed out by Ward (1945), linear potential theory may provide solutions with singularities, thus leading to non-uniformly converging series which are unsuitable for direct numerical computation. For the case of harmonic azimuthal dependence of the velocity potential, it has been shown in § 3 that this phenomenon arises whenever abrupt bending of the streamlines and thus, in physical reality, shock or expansion waves occur, which linear potential theory is unable to describe correctly. By means of the decomposition (3.13), the singular parts of any such solution can be expressed in terms of the first derivatives of the universal series (3.11), which themselves can be evaluated by an analytical method based on Kummer's series transformation. For a broad class of solutions including the general axisymmetric case, the difficulties preventing the practical application of linear potential theory have thus been resolved.

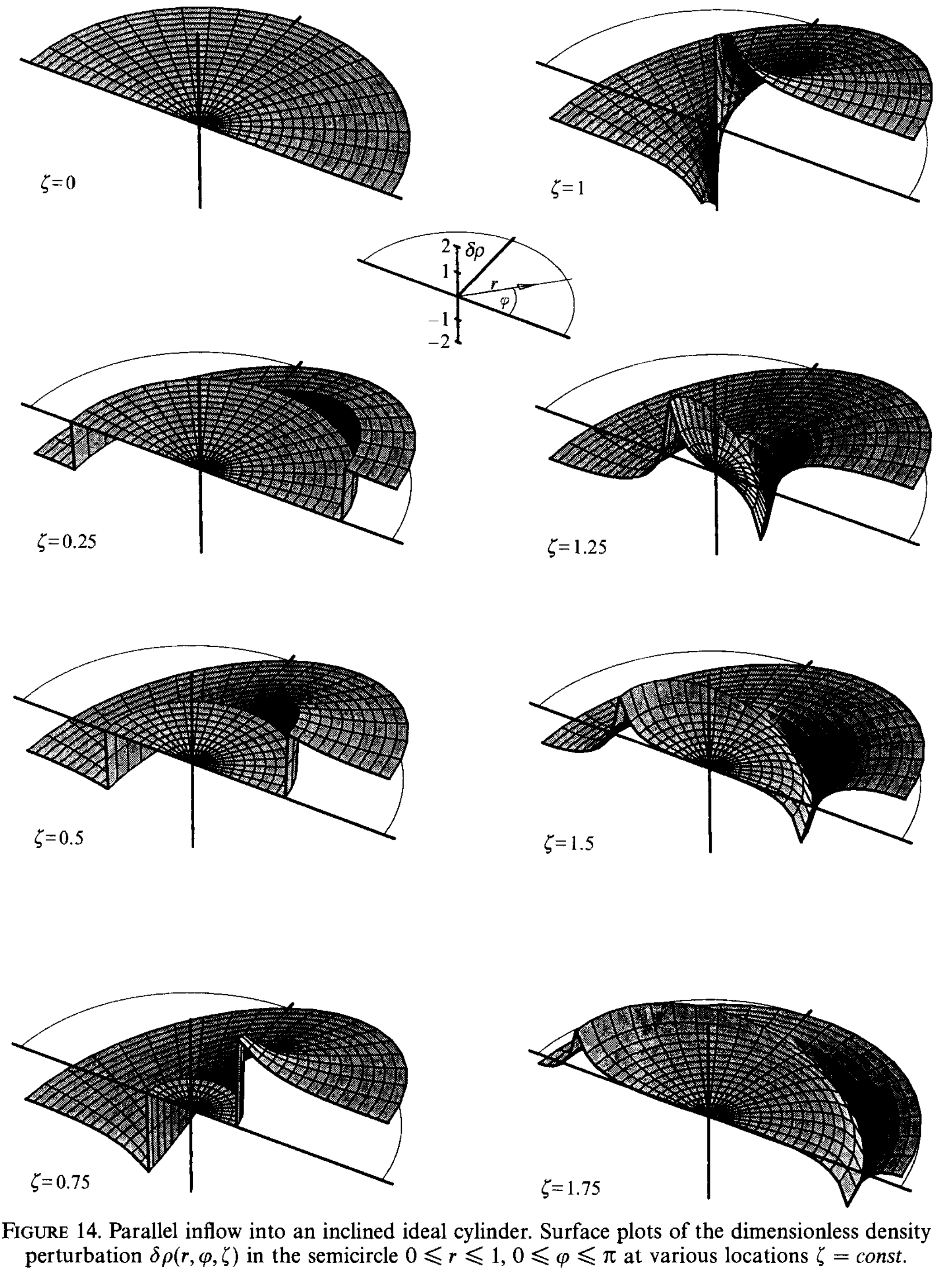

FIGURE 14. Parallel inflow into an inclined ideal cylinder. Surface plots of the dimensionless density perturbation $\delta\rho(r,\varphi,\zeta)$ in the semicircle $0 \leqslant r \leqslant 1$, $0 \leqslant \varphi \leqslant \pi$ at various locations $\zeta = const.$

For a qualitative understanding of the resulting flow fields, the asymptotic laws governing the propagation of small perturbations along the characteristics of the wave equation have been derived in § 4. To a first order of approximation, perturbations of axial velocity (and thus pressure, density and temperature) are reflected without phase shift at the wall, whereas the transmission of the axis results in a phase shift of $\pi/2$, thus changing the symmetry properties of a perturbation with respect to the characteristic. Similarly, while a perturbation of radial velocity is reflected at the wall with a phase shift of π, it suffers a phase shift of $-\pi/2$ on crossing the axis of the duct. While the asymptotic reflection law could be intuitively expected since the same result holds exactly in the case of plane flow, no analogy exists for the asymptotic transmission law. In addition, perturbations increase asymptotically as $r^{-1/2}$ as the axis is approached, since on travelling from the wall to the axis on the surface of a Mach cone, the perturbations are focussed into a single point.

Some elementary cases of axisymmetric and non-axisymmetric flows have been discussed in § 5. For the case of a discontinuity in wall slope, the flows exhibit a characteristic cellular pattern which is strongly dominated by the periodically distributed singularities and regions of extreme magnitudes of the flow variables in the vicinity of the focus points on the axis, whereas for smooth wall contours, the flow variables are bounded and continuous throughout the field and singularities occur only in their higher derivatives.

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